Applications of Optical Homodyne Tomography

Jinwei Wu

B. Sc., University of Science and Technology of China, 1992.
M. Sc., University of Science and Technology of China, 1995.

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Declaration

This thesis is an account of research undertaken between June 1996 and June 1999 in *the Department of Physics, Faculty of Science, Australian National University, Canberra, Australia.*

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

Jinwei Wu
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Abstract

This thesis discusses optical homodyne tomography (OHT), a powerful new technique for quantum state reconstruction (QSR). In contrast to conventional quantum-optical measurement, such as direct intensity measurement and standard homodyne detection, OHT can provide complete information of the quantum state, in terms of the reconstructed Wigner function.

A semiclassical theory, developed to model the OHT system, is presented. No approximation was made for the quantum state itself so this model can preserve any non-Gaussian feature of the state being measured. An asynchronous demodulation scheme was included in the modeling and phase quadrature diffusion in phase space was predicted in this case.

Numerical methods were implemented for data acquisition/binning and the inverse Radon transform, the latter is for Wigner function reconstruction. Combining this with a sophisticated hardware configuration, and precise optical phase extraction, it was possible to investigate some classical and quantum states. As an alternative method for doing the inverse Radon transform, an integral was developed using the degenerate hypergeometric function.

We used OHT to measure the amplitude and phase noise of a diode pumped Nd:YAG laser. This phase sensitive technique together with a high finesse optical cavity configuration, gave us the full noise spectra for both the amplitude and phase noise. Elliptic super-Poissonian Wigner functions were reconstructed and the phase noise was found to dominate in most radio frequency (RF) regimes. An unknown signal embedded in the laser system was found to cause phase quadrature diffusion.

By means of modulation-switching and phase-unlocking, classical non-Gaussian states were produced and their Wigner functions reconstructed. Two-peak and phase quadrature diffused Wigner functions were observed. The ability of OHT to simultaneously measure the signal and noise of an unknown state was demonstrated. It is shown to be an effective technique for measuring the signal to noise ratio (SNR) in any quadrature.

OHT was used to measure a squeezed vacuum state and visualise the squeezing effect in phase space. Detailed analyses of the reconstructed Wigner function and the measured probability density functions (PDF's) were made and compared. Some extra noise sources were identified and their effects discussed. Higher order moments were calculated and the ability and limitations of realistic OHT assessed.

An overview of possible techniques for producing quantum non-Gaussian states was made and a method to generally calculate the Wigner functions of the output states of self-phase modulation and cross-phase modulation presented.

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Abbreviations

AM	Amplitude modulation
BS	Beam splitter
EOM	Electro-optic modulator
FSR	Free spectral range
FWHM	Full-width-half-maximum
LO	Local oscillator
LPF	Low pass filter
NPBS	Non-polarising beam splitter
NPRO	Non-planar ring oscillator
OHT	Optical homodyne tomography
OPO	Optical parametric oscillator
PBS	Polarising beam splitter
PD	Photodetector
PDF	Probability density function
PM	Phase modulator
PZT	Piezo-electric transducer
QNL	Quantum noise limit
QPD	Quasiprobability distribution
QSR	Quantum state reconstruction
RBW	Resolution bandwidth
RF	Radio frequency
RRO	Resonant relaxation oscillation
SHG	Second harmonic generation
SNR	Signal to noise ratio
VBW	Video bandwidth
VCA	Voltage controlled attenuator

The Tao (Way) that can be told of is not the eternal Tao;
The name that can be named is not the eternal name.
The Nameless is the origin of Heaven and Earth;
The Named is the mother of all things.
Therefore let there always be non-being so we may see their subtlety,
And let there always be being so we may see their outcome.
The two are the same,
But after they are produced, they have different names.
They both may be called deep and profound (hsüan).
Deeper and more profound,
The door of all subtleties!

Tao-te ching[1], Lao Tzu

道 [道]

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[1]English translation by Wing-Tsit Chan, in *A source book in Chinese philosophy*, Princeton university press, New Jersey, 1963.

Chapter 1

Introduction

1.1 Overview of quantum state reconstruction

Classical physics was a beautiful dream of our universe. In the era of classical physics, everything seemed predictable and deterministic. As long as you knew the initial condition of an isolated system, using Newton's law, you could precisely predict all future behaviour of that system. Indeed, classical physics works incredibly well in the macroscopic world. However, at the beginning of the twentieth century, this dream was terminated by quantum mechanics. This new theory was discovered when people tried to explain some microscopic phenomena. Planck discovered a new law of black body radiation and explained this law by a new concept — quanta [2, 3]. Einstein used the concept of a photon to explain the photoelectric effect [4]. Later on modern quantum mechanics emerged and is now the standard theory of all microscopic phenomena. Classical physics turns out to be the good enough approximation of quantum mechanics for the macroscopic world. According to quantum mechanics, we can not know the initial condition precisely. For example, when we know the position of a particle precisely, we lose all the information about its momentum. There is always a certain amount of uncertainty in characterising an object. This is the famous *Heisenberg uncertainty principle*. This uncertainty is of great importance in quantum optics, especially in squeezing experiments.

In the classical description of an object, if we simultaneously measure the position and momentum of this object, we characterise the state of this object completely. Now, according to quantum mechanics, this is no longer possible. An object is fully described by its wave function. If we want to fully describe a quantum object, we have to know its wave function, or some other equivalent information. This is the idea of quantum state reconstruction (QSR). However, the wave function itself is normally not measurable. As we will discuss in chapter 2, QSR is done by reconstructing the quasiprobability distributions (QPD's) or density matrices of quantum states. QPD's and density matrices are equivalent information to the quantum wave functions.

Bertrand and Bertrand [5] introduced tomography into quantum mechanics and pointed out a way to reconstruct the Wigner functions (an important QPD, see chapter 2) from marginals. However, the many works of QSR in quantum optics have been inspired by another pioneering paper of Vogel and Risken [6]. They described a way to reconstruct QPD's from the measured quadrature amplitude distributions by optical homodyne measurement. The first experiment of QSR was carried out by Smithey, et al. [7–9] in M. G. Raymer's group. The Wigner functions of a vacuum and a quadrature-squeezed state were measured. In their implementation of balanced homodyne detection, a pulsed signal field E_s was superposed at a 50:50 beam splitter with a pulsed coherent-state field E_{LO} and they called this new technique *optical homodyne tomography* (OHT). Later on, OHT was proposed and initiated by U. Leonhardt [10] and carried out for a continuous wave light field by Breitenbach et al. [11, 12]. The Wigner functions of a continuous-wave, strongly squeezed vacuum state and the squeezed coherent states were reconstructed in

their experiment. From then on, OHT has been carried out in several experiments [13–17] and also used to measure the phase noise of a diode pumped Nd:YAG laser [18, 19]. Besides optical state reconstruction, there are also reports of QSR for other quantum systems. The quantum state of a molecular vibrational mode was reconstructed using fluorescence tomography by Dunn et al. [20]; the motional quantum state of a single ion in an RF Paul's trap was reconstructed by Leibfried et al. [21]; and the Wigner function of an ensemble of helium atoms was reconstructed by Kurtsiefer et al. [22].

An alternative method to the Wigner function reconstruction is to reconstruct the density matrix of the quantum state in Fock representation [12, 21, 23–25]. This directly provides the density matrix elements $\rho_{mn} \equiv \langle m|\hat{\rho}|n\rangle$ of the state. This method is regarded as a more direct way for calculating expectation values of variables. However in optics, we find that if we want to visualise non-classical light, such as the squeezing ellipse, it is better to reconstruct the Wigner function itself[1].

A second alternative proposal for reconstructing the quantum states of light is the so-called direct probing method [26–28] which uses a beam splitter to mix the signal beam and a strong local oscillator. By choosing the transmittance of the signal beam close to unity and varying the amplitude and phase of the local oscillator, the Wigner function of the signal beam can be reconstructed in phase space point by point without the use of integration (please refer to Appendix B for a description of this method). This elegant and simple method was also experimentally demonstrated by Banaszek et al. [29]. However, the quantum efficiency of photon counters is normally much lower than that of photodiodes used in homodyne measurement, and the extra noise of the local oscillator can not be canceled out as in a balanced homodyne scheme. These are clear drawbacks of the direct probing scheme. However, it has been experimentally demonstrated by Munroe et al. [30] that photon statistics can be retrieved from the phase-averaged quadrature-field distribution. This can solve the problem of low quantum efficiency of photon counting. Inspired by this method, another proposal, i. e., the so-called cascaded optical homodyning [31] was developed. This cascaded method combines the simplicity of local sampling and the high quantum efficiency of balanced homodyning. It has not been experimentally demonstrated yet.

In the area of QSR, there are many excellent theoretical developments. A good summary can be found in the book by U. Leonhardt [32]. Here we want to mention some of them. It was found that the density matrix can be reconstructed via pattern functions [33–42]. Leonhardt and Raymer proposed a way to infer the quantum state of a wave packet from its position probability distributions [43]. Twin-beam [44] and multimode [45–48] QSR schemes were also proposed. Finally, in the field of cavity QED, where highly non-classical microwave fields can be generated (see section 9.1), beautiful theoretical schemes have been proposed using magnetic tomography [49] or microwave seeding [50, 51]. We note that the microwave seeding method is analogous to the direct probing method [26–28] for optical state reconstruction. The relatively higher detection efficiency of atoms in cavity QED makes this method very attractive. Reconstruction of an entangled state in cavity QED was also proposed [52].

In conclusion, QSR is still a fast growing area of quantum optics and we should be able to see some more exciting experimental and theoretical progress in the near future.

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[1]Note that this statement is true for continuous variables. For discrete variables, such as polarisation of photons, we need to reconstruct the density matrix in the appropriate basis.

1.2 Motivation

The phase noise of a laser is of concern when we want to apply phase or frequency modulation and carry out measurements of these properties. However, most investigations have been limited to the measurements of the linewidths of lasers. As a phase sensitive technique, how can OHT help us in determining the full phase noise of a laser? We used OHT and a high finesse cavity configuration to measure the phase noise spectrum as well as the intensity noise spectrum of a diode pumped Nd:YAG laser.

The real strength of OHT is to reveal the non-Gaussian features of quantum and classical states and several fundamental and technical questions have to be addressed: To what extent OHT reveals the non-Gaussian features of unknown states? What will be the requirements for the electronic configuration as well as the numerical reconstruction? How will modulation signals be represented in phase space? How will an asynchronous demodulation scheme affect the result of OHT? This leads to our systematic investigation of modulated coherent states where we used OHT to reconstruct their Wigner functions. Different modulation and demodulation schemes were demonstrated.

Besides the measurements of classical states, questions arise on how OHT can help to determine a quantum state. Will there be any difference between a classical state OHT and a quantum state OHT? What are the capability and limitations of OHT for a quantum state measurement? These points are answered through our experiments with squeezed light where we used OHT to measure non-classical Wigner functions. As an example, we used OHT to measure a squeezed vacuum state. Comparison between the result of OHT and the conventional homodyne measurement was made and some extra noise sources were identified and their effect discussed.

1.3 Thesis structure

An introduction to the basic phase space theory of quantum mechanics is given in chapter 2. Some theoretical concepts for characterising quantum mechanical states will be explained. In particular, we introduce the Wigner function, an important quasiprobability distribution of quantum states. We summarise different quantum-optical measurements in chapter 3. OHT will be introduced in this chapter and a semiclassical model of the OHT system will be provided. In chapter 4, we focus on the numerical implementation of OHT. Data acquisition, binning and numerical realisation of the inverse Radon transform will be discussed in detail. We also present our numerical simulation of quantum state reconstruction in this chapter. Experimental details of the optics and electronics used as well as the electronic configuration of OHT are discussed in chapter 5. In particular, we outline the formulas for optical cavity calculations, because optical cavities were used in our laser noise measurement experiment. Chapter 6 is devoted to laser phase noise measurement. We will discuss the difficulty of measuring the phase noise of a laser and present our high finesse cavity configuration. We demonstrate the ability of OHT for revealing the non-Gaussian features of unknown input states in chapter 7. Classical non-Gaussian states were generated and reconstructed. Chapter 8 is about the application of OHT to quantum state measurement. The state that will be investigated is a squeezed vacuum state. In chapter 9 we will overview the progress and difficulty of producing quantum non-Gaussian states and present our calculation for self-phase modulation and cross-phase modulation. Finally in chapter 10 we will conclude this thesis and discuss the future of OHT. The Appendices include the C programs for data binning and the

inverse Radon transform.

Chapter 2

Basic phase space theory of quantum mechanics

2.1 Introduction

In this chapter, we will discuss some basic concepts of quantum mechanics for characterising quantum states in the context of quantum optics. The focus will be the phase space method because optical homodyne tomography (OHT) of quantum states is a phase space reconstruction. First we will introduce some states of light and their quantum features. Then we will introduce the concept of density operator in section 2.3. This leads to the definition of quasiprobability distributions (QPD's) and s-parameterisation in section 2.4. Finally we will outline the properties of the Wigner function, an important QPD of quantum states.

2.2 States of light

In quantum mechanics, we can fully describe a quantum state by a wave function (also called state vector). There are some common quantum states of light in quantum optics which we want to summarise here. First we note that there are two kinds of light field, one is confined in an optical cavity (i. e., standing waves), the other one is a traveling wave light field. We don't want to distinguish them in our general discussion.

(a) Fock state $|n\rangle$.
These are the eigenstates of the photon number operator $\hat{n} = \hat{a}^\dagger \hat{a}$,

$$\hat{n}|n\rangle = n|n\rangle. \tag{2.1}$$

Here $\hat{a}$ is the annihilation operator of the optical field and n is the eigenvalue (photon number) of the photon number operator. Fock states correspond to the eigenstates of harmonic oscillators. As we will discuss in section 2.4 and section 2.5, there are negative values in their Wigner functions, which is obviously a non-classical feature. Fock state of the motion of a trapped atom was generated experimentally [21, 53]. Recently Fock states of the micromaser field were generated in microwave cavity QED experiment [54].

(b) Coherent state $|\alpha_0\rangle$.
The concept of coherent state was first introduced by Glauber [55]. It's the eigenstate of the annihilation operator $\hat{a}$,

$$\hat{a}|\alpha_0\rangle = \alpha_0|\alpha_0\rangle, \tag{2.2}$$

where α_0 is simply a parameter in the complex plane. This concept turns out to be a very useful tool in quantum physics. The light field emitted by an ideal laser is normally

regarded as a coherent state. In practice only the high RF modulated sidebands of a laser are treated as coherent states, because the amplitude and phase noise of a laser approach quantum noise limit in this RF regime. Coherent state $|\alpha_0\rangle$ can be expanded in Fock representation as

$$|\alpha_0\rangle = e^{-|\alpha_0|^2/2} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} |n\rangle. \tag{2.3}$$

(c) Vacuum state $|0\rangle$.
The vacuum state can be regarded as a special case of the coherent state where the coherent amplitude $\alpha_0 = 0$[1]. It is everywhere in our universe and forms a continuous spectrum. There is an energy $\frac{1}{2}\hbar\omega$ associated with the vacuum state at a frequency of ω. This is the so-called *zero point energy*.

(d) Squeezed coherent state $\hat{D}(\alpha_0)\hat{S}(\zeta)|0\rangle$.
This state can be regarded as the squeezed and displaced vacuum state.

$$\hat{S}(\zeta) \equiv \exp\left[\frac{1}{2}\zeta^*\hat{a}^2 - \frac{1}{2}\zeta(\hat{a}^\dagger)^2\right] \tag{2.4}$$

is the squeezing operator where $\zeta = re^{i\theta}$ is the squeezing parameter. This operator has the effect of squeezing one quadrature and makes this quadrature quieter while the orthogonal quadrature noisier.

$$\begin{aligned} \hat{D}(\alpha_0) &\equiv \exp(\alpha_0\hat{a}^\dagger - \alpha_0^*\hat{a}) \\ &= e^{-|\alpha_0|^2/2}e^{\alpha_0\hat{a}^\dagger}e^{-\alpha_0^*\hat{a}} \end{aligned} \tag{2.5}$$

is the displacement operator. It can shift the quantum state away from the origin of phase space and induce a coherent amplitude α_0[2]. The experimentally quoted degree of squeezing x [dB] can be evaluated as

$$x = 20r/\ln(10). \tag{2.6}$$

A squeezed coherent state is also called a bright squeezed state because of the coherent amplitude α_0.

(e) Squeezed vacuum state $\hat{S}(\zeta)|0\rangle$.
This is a special case of (d) where $\alpha_0 = 0$, i. e. , there is no displacement operation after the squeezing operation. A squeezed vacuum state can be generated by an optical parametric oscillator. This was first achieved by Wu et al. [56].

(f) Chaotic state $\hat{\rho} = (1 - e^{-\beta}) \sum_{n=0}^{\infty} |n\rangle\langle n|e^{-n\beta}$.
A chaotic state is also called a thermal state. It's a mixture state so we can only write down its density operator (as we will discuss in the next section). For a system in thermal

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[1]It can also be regarded as a special case of the Fock state $|n\rangle$, where $n = 0$.

[2]Squeezed coherent state can also be defined as $\hat{S}(\zeta)\hat{D}(\alpha_0)|0\rangle$ which corresponds to what happens physically, i. e. a bright coherent state $\hat{D}(\alpha_0)|0\rangle$ is squeezed. It can be proved that $\hat{S}(\zeta)\hat{D}(\alpha_0)|0\rangle = \hat{D}(\beta)\hat{S}(\zeta)|0\rangle$ where $\beta = \alpha_0\cosh(r) + \alpha_0^*e^{i\theta}\sinh(r)$.

equilibrium,

$$\langle N \rangle = \frac{1}{e^{\beta} - 1} \tag{2.7}$$

is the average photon number. Here $\beta = \hbar\omega/k_B T$ and T is the temperature of the system.

(g) "Schrödinger's cat" state.
These kinds of states are named after Schrödinger's famous Gedanken experiment where the quantum superposition is transferred to a macroscopic level so that a "quantum cat" can be alive and dead at the same time [57]. A "Schrödinger's cat" state can be defined as the coherent superposition of distinguishable macroscopic states[3]. For optical fields, a "Schrödinger's cat" state can be the superposition of distinguishable coherent states such as [6, 58]

$$|\psi\rangle = \frac{|a + ib\rangle + |a - ib\rangle}{\{2 \left[1 + \cos(2ab) \exp(-2b^2) \right]\}^{1/2}}. \tag{2.8}$$

The even and odd "Schrödinger's cat" states are superpositions of coherent states with opposite phase

$$|\alpha\rangle_e = \frac{1}{2} e^{|\alpha|^2/2} (\cosh(|\alpha|^2))^{-1/2} (|\alpha\rangle + |-\alpha\rangle), \tag{2.9}$$

$$|\alpha\rangle_o = \frac{1}{2} e^{|\alpha|^2/2} (\sinh(|\alpha|^2))^{-1/2} (|\alpha\rangle - |-\alpha\rangle). \tag{2.10}$$

They are eigenstates of the squared annihilation operator $\hat{a}^2$ [59], i. e.

$$\begin{aligned} \hat{a}^2 |\alpha\rangle_e &= \alpha^2 |\alpha\rangle_e, \\ \hat{a}^2 |\alpha\rangle_o &= \alpha^2 |\alpha\rangle_o. \end{aligned} \tag{2.11}$$

Experimentally these kinds of states are extremely hard to generate because of quantum decoherence [60]. Nevertheless, there are experimental investigations along this line. Monroe et al. [61] generated a "Schrödinger's cat" superposition state of an atom. A trapped $^9Be^+$ ion was laser-cooled to the zero point energy and then prepared in a superposition of spatially separated coherent harmonic oscillator states. Brune et al. created a mesoscopic superposition of quantum states involving radiation fields with classically distinct phases and observed its progressive decoherence [62].

2.3 Density operator

It's not always possible for us to write down a wave function for a system. Suppose there is an ensemble of particles, each of the particles can be in either state $|a\rangle$ or $|b\rangle$ (but only one of them). We only know that the classical probability of finding it in state $|a\rangle$ or $|b\rangle$ are both 50%. In this case, the system is in a classical mixture state of $|a\rangle$ and $|b\rangle$. This is very different from the coherent superposition of $|a\rangle$ and $|b\rangle$. In order to describe this particle, we need the density operator

$$\hat{\rho} = \frac{1}{2} |a\rangle\langle a| + \frac{1}{2} |b\rangle\langle b|. \tag{2.12}$$

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[3]In Schrödinger's Gedanken experiment there was entanglement between one microscopic superposition and another macroscopic superposition. It is this entanglement that is utilised to create the "cat" state.

The density operator is more general than the wave function. For a pure state, we can write down a wave function $|\psi\rangle$, and the corresponding density operator is

$$\hat{\rho} = |\psi\rangle\langle\psi|. \tag{2.13}$$

We can easily distinguish between a pure state and a mixture state using the density operator. For a pure state,

$$\hat{\rho}^2 = \hat{\rho} \tag{2.14}$$

and the trace of the squared density operator is 1,

$$Tr(\hat{\rho}^2) = Tr(\hat{\rho}) = 1. \tag{2.15}$$

Whilst for a mixture state

$$\hat{\rho}^2 \neq \hat{\rho} \tag{2.16}$$

and

$$Tr(\hat{\rho}^2) < 1. \tag{2.17}$$

In this sense $Tr(\hat{\rho}^2)$ is a measure of the purity of a system. (We note that $Tr(\hat{\rho}) = 1$ is true for both a pure state and a mixture state.)

The complete information of a state is contained by a density operator. If we can somehow reconstruct the density operator in a representation, we know everything of the quantum state already. In Fock representation, we need to reconstruct the matrix elements $\rho_{mn} \equiv \langle m|\hat{\rho}|n\rangle$. This is one way of quantum state reconstruction [12, 21, 23–25].

2.4 Quasiprobability distributions and s-parameterisation

As discussed in the previous section, the density operator $\hat{\rho}$ contains the full information about a quantum state. On the other hand, the only information we can ask about a quantum state are the expectation values of observables. Given any observable $\hat{F}$ of a quantum state, the expectation value of $\hat{F}$ is the trace of the product of the operators $\hat{\rho}$ and $\hat{F}$

$$\langle\hat{F}\rangle = Tr(\hat{\rho}\hat{F}). \tag{2.18}$$

To simplify the evaluation of this trace of product, a number of procedures have been introduced. $\langle\hat{F}\rangle$ can be written as an integral of the product of a weight function $w(\alpha)$ which refers to the quantum state, and a function $f(\alpha)$ which refers to the operator $\hat{F}$

$$\langle\hat{F}\rangle = \int w(\alpha)f(\alpha)d^2\alpha. \tag{2.19}$$

This is the so-called phase space method, in the sense the integration is carried out over the complex α plane. Eq. (2.19) is a simple and elegant realisation of Eq (2.18). It is analogous to the classical phase space method, where $w(\alpha)$ is the classical probability distribution function. However, for a quantum state, $w(\alpha)$ is in general not interpretable as a probability distribution function. Rather, $w(\alpha)$ can take on negative values. We call it quasiprobability distribution (QPD).

QPD's such as the Glauber-Sudarshan P function [55, 63], the Wigner function [64] and the Q function [55, 63] are very important QPD's in quantum optics. We can calculate the expectation value of any product of creation operators $\hat{a}^\dagger$ and annihilation operators

$\hat{a}$ by using these functions.

(a) P function.
The P function is a diagonal representation[4] of the density operator $\hat{\rho}$

$$\hat{\rho} = \int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha. \tag{2.20}$$

If we define the normally ordered characteristic function $\chi_N(\xi)$ as the expectation value

$$\chi_N(\xi) = Tr[\hat{\rho}\exp(\xi\hat{a}^\dagger)\exp(-\xi^*\hat{a})], \tag{2.21}$$

then we can verify that the weight function $P(\alpha)$ is the complex Fourier transform of $\chi_N(\xi)$,

$$P(\alpha) = \pi^{-2}\int \exp(\alpha\xi^* - \alpha^*\xi)\chi_N(\xi)d^2\xi. \tag{2.22}$$

The expectation value of the normally ordered product $(\hat{a}^\dagger)^n\hat{a}^m$ can be written as

$$\langle(\hat{a}^\dagger)^n\hat{a}^m\rangle = \int (\alpha^*)^n\alpha^m P(\alpha)d^2\alpha. \tag{2.23}$$

(b) Wigner function.
Similarly, we can define the symmetrically ordered characteristic function

$$\begin{aligned}\chi(\xi) &= Tr[\hat{\rho}\hat{D}] \\ &= Tr[\hat{\rho}\exp(\xi\hat{a}^\dagger - \xi^*\hat{a})],\end{aligned} \tag{2.24}$$

then the Wigner function $W(\alpha)$ is again the complex Fourier transform of $\chi(\xi)$,

$$W(\alpha) = \pi^{-2}\int \exp(\alpha\xi^* - \alpha^*\xi)\chi(\xi)d^2\xi. \tag{2.25}$$

The Wigner function $W(\alpha)$ is a well-behaved function and exists for any quantum state. It is a bounded function

$$|W(\alpha)| \le 2/\pi. \tag{2.26}$$

The expectation value of the symmetrically ordered product $\{(\hat{a}^\dagger)^n\hat{a}^m\}_0$ is given by the integral

$$\langle\{(\hat{a}^\dagger)^n\hat{a}^m\}_0\rangle = \int W(\alpha)(\alpha^*)^n\alpha^m d^2\alpha. \tag{2.27}$$

(c) Q function.
We can also define the antinormally ordered characteristic function

$$\chi_A(\xi) = Tr[\hat{\rho}\exp(-\xi^*\hat{a})\exp(\xi\hat{a}^\dagger)], \tag{2.28}$$

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[4]If the density operator $\hat{\rho}$ is expanded into $|\alpha\rangle\langle\alpha'|$, where α and α' are two independent parameters, then we get the non-diagonal representation of $\hat{\rho}$.

then the Q function is the complex Fourier transform of $\chi_A(\xi)$,

$$Q(\alpha) = \pi^{-2} \int \exp(\alpha\xi^* - \alpha^*\xi)\chi_A(\xi)d^2\xi. \tag{2.29}$$

The Q function is also a well-behaved function and exists for any quantum state. The expectation value of the antinormally ordered product $\hat{a}^m(\hat{a}^\dagger)^n$ can be written as

$$\langle\hat{a}^m(\hat{a}^\dagger)^n\rangle = \int (\alpha^*)^n\alpha^m Q(\alpha)d^2\alpha. \tag{2.30}$$

$Q(\alpha)$ is non-negative and bounded between 0 and $1/\pi$

$$0 \le Q(\alpha) \le 1/\pi. \tag{2.31}$$

In Table 2.1 we list the P, Wigner and Q functions of some common quantum and classical states. Notice the definitions of the complex parameters

$$\begin{aligned}
\alpha &\equiv& \alpha_r + i\alpha_i, \\
\alpha_{0r} &\equiv& \alpha_{0r} + i\alpha_{0i}, \\
\beta &\equiv& \beta_r + i\beta_i.
\end{aligned} \tag{2.32}$$

For Fock state $|n\rangle$, the Wigner function is proportional to the Laguerre polynomial $L(4|\alpha|^2)$. Here are some examples of the Laguerre polynomials,

$$\begin{aligned}
L_0(x) &=& 1, \\
L_1(x) &=& 1 - x, \\
L_2(x) &=& 1 - 2x + \frac{x^2}{2}.
\end{aligned} \tag{2.33}$$

Cahill and Glauber [65] introduced s-parameterised QPD's. They defined the s-ordered characteristic function

$$\chi(\xi, s) = Tr[\hat{\rho}\hat{D}(\xi, s)], \tag{2.34}$$

where

$$\begin{aligned}
\hat{D}(\xi, s) &=& \hat{D}(\xi)e^{s|\xi|^2/2} \\
&=& \exp(\xi\hat{a}^\dagger - \xi^*\hat{a} + s|\xi|^2/2)
\end{aligned} \tag{2.35}$$

is the s-ordered displacement operator. The s-parameterised QPD is the complex Fourier transform of $\chi(\xi, s)$ (we note the deviation by a factor of $1/\pi$ from the definitions of Ref. [65]),

$$W(\alpha, s) = \pi^{-2} \int \chi(\xi, s)\exp(\alpha\xi^* - \alpha^*\xi)d^2\xi. \tag{2.36}$$

In this format, $W(\alpha, 1)$ is the Glauber-Sudarshan P function, $W(\alpha, 0)$ is the Wigner function, and $W(\alpha, -1)$ is the Q function. Any single-time expectation value of the s-ordered products $\langle[(\hat{a}^\dagger)^n\hat{a}^m]_s\rangle$ can be obtained by proper integration with weight $W(\alpha, s)$ in the complex α plane.

There is a convolution law between the s-parameterised QPD's

$$W(\alpha, s) = \frac{2}{t-s} \int \exp\left(-\frac{2|\alpha - \beta|^2}{t-s}\right) W(\beta, t) \frac{d^2\beta}{\pi}, \tag{2.37}$$

where $s < t$. Here we give some special cases. The Wigner function is the Gaussian convolution of the P function,

$$W(\alpha) = \frac{2}{\pi} \int P(\beta) \exp(-2|\beta - \alpha|^2) d^2\beta. \tag{2.38}$$

The Q function is the Gaussian convolution of the Wigner function or the P function,

$$Q(\alpha) = \frac{2}{\pi} \int W(\beta) \exp(-2|\beta - \alpha|^2) d^2\beta, \tag{2.39}$$

$$Q(\alpha) = \frac{1}{\pi} \int P(\beta) \exp(-|\beta - \alpha|^2) d^2\beta. \tag{2.40}$$

Due to the convolution law, lower-ordered QPD's are smoother than higher-ordered QPD's.

It can be seen from the theory that there is a one-to-one correspondence between the QPD and the density operator. If we can measure the QPD of a quantum state, this quantum state is completely characterised. Vogel and Risken [6] pointed out that one can obtain the Wigner distribution by tomographic inversion of a set of measured probability distributions, $P_\theta(x_\theta)$, of the quadrature amplitudes. Their pioneering work inspired the fast development in this area. We show in Fig. 2.1 the link between some concepts discussed in this chapter for characterising quantum states.

Figure 2.1: From experimental data to the complete characterisation of a quantum state. OHT: optical homodyne tomography.

2.5   Wigner function

First discovered by Wigner in 1932 [64] when he tried to make quantum correction for thermodynamic equilibrium, the Wigner function is now an useful tool in many fields of physics. It can be a classical joint-distribution as well as a quantum QPD. It turns out to be an elegant and simple method of overlap calculation. In quantum optics, as pointed out by Ulf Leonhardt[32], the Wigner function is of particular importance. It is the closest analogue to classical probability distributions. This can be seen from many situations, some of which we will discuss below. Theoretically, all of the QPD's are equivalent. However, lower-ordered QPD's such as the Q function are too smooth and have hidden many details which can be displayed by the Wigner function. Practically it's not realistic to retrieve higher-ordered QPD's from lower-ordered QPD's, because deconvolution is very sensitive to errors.

Here we summarise some properties of the Wigner function. The Wigner function is real,

$$W^*(\alpha) = W(\alpha). \tag{2.41}$$

The integration of the Wigner function in phase space is 1,

$$\int W(\alpha)d^2\alpha = 1. \tag{2.42}$$

The Wigner function is bounded,

$$|W(\alpha)| \leq 2/\pi. \tag{2.43}$$

There are elegant overlap formulas for the Wigner function. The expectation value of the product of two operators is the overlap of their Wigner functions,

$$Tr\{\hat{F}_1\hat{F}_2\} = \pi \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W_{F1}(\alpha)W_{F2}(\alpha)d^2\alpha. \tag{2.44}$$

The expectation value of an operator is the overlap of its Wigner function and the Wigner function of the quantum state,

$$Tr\{\hat{\rho}\hat{F}\} = \pi \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W(\alpha)W_F(\alpha)d^2\alpha. \tag{2.45}$$

The square of the inner product of two quantum states is the overlap of their Wigner functions,

$$|\langle\psi_1|\psi_2\rangle|^2 = \pi \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W_1(\alpha)W_2(\alpha)d^2\alpha. \tag{2.46}$$

The trace of the squared density operator is the self-overlap of its Wigner function,

$$Tr\{\hat{\rho}^2\} = \pi \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W(\alpha)^2 d^2\alpha. \tag{2.47}$$

Here we have used the definition of the Wigner function of an arbitrary operator $\hat{F}$ [65, 66]

$$W_F(\alpha) = \frac{1}{\pi}Tr[\hat{F}\hat{T}(\alpha,0)], \tag{2.48}$$

where $\hat{T}(\alpha, s)$ (in Eq. (2.48), $s = 0$) are defined as the complex Fourier transforms of the

s-parameterised displacement operators $\hat{D}(\alpha, s)$ [65, 66],

$$\hat{T}(\alpha, s) = \int \hat{D}(\xi, s) \exp(\alpha\xi^* - \alpha^*\xi) \frac{d^2\xi}{\pi}. \tag{2.49}$$

The only pure quantum states that don't have negative Wigner functions are those states having Gaussian wave functions [67–69]. We call these pure Gaussian states. Examples of Gaussian states are the coherent states and the squeezed coherent states (see section 2.2). By this definition, all non-Gaussian states have negative value for their Wigner functions. We show in Fig. 2.2 and Fig. 2.3 the Wigner functions of some quantum states.

2.6 Summary

In this chapter we introduced the basic phase space theory of quantum mechanics in the context of quantum optics. We explained some common quantum states of light field and discussed their properties. The density operator was introduced and the way to distinguish between a pure state and a mixture state presented. s-parameterisation of QPD's discussed and the P, Wigner and Q functions of some quantum states summarised. Finally we focused on the Wigner function and discussed its importance in quantum state reconstruction.

Coherent state $\|\alpha_0\rangle$	$P(\alpha)$	$\delta^{(2)}(\alpha - \alpha_0)$
	$W(\alpha)$	$2\exp(-2\|\alpha - \alpha_0\|^2)/\pi$
	$Q(\alpha)$	$\exp(-\|\alpha - \alpha_0\|^2)/\pi$
Vacuum state $\|0\rangle$	$P(\alpha)$	$\delta^{(2)}(\alpha)$
	$W(\alpha)$	$2\exp[-2\|\alpha\|^2]/\pi$
	$Q(\alpha)$	$e^{-\|\alpha\|^2}/\pi$
Squeezed coherent state $\hat{D}(\alpha_0)\hat{S}(\zeta)\|0\rangle$ ($\zeta = r$ is real)	$P(\alpha)$	—
	$W(\alpha)$	$2\exp\{-2[(\alpha_r - \alpha_{0r})^2/e^{-2r} + (\alpha_i - \alpha_{0i})^2/e^{2r}]\}/\pi$
	$Q(\alpha)$	$\exp\{-2[\frac{(\alpha_r-\alpha_{0r})^2}{e^{-2r}+1} + \frac{(\alpha_i-\alpha_{0i})^2}{e^{2r}+1}]\}/(\pi\cosh r)$
Squeezed vacuum state $\hat{S}(\zeta)\|0\rangle$ ($\zeta = r$ is real)	$P(\alpha)$	—
	$W(\alpha)$	$2\exp\{-2[\alpha_r^2/e^{-2r} + \alpha_i^2/e^{2r}]\}/\pi$
	$Q(\alpha)$	$\exp\{-2[\frac{\alpha_r^2}{e^{-2r}+1} + \frac{\alpha_i^2}{e^{2r}+1}]\}/(\pi\cosh r)$
Fock state $\|n\rangle$	$P(\alpha)$	—
	$W(\alpha)$	$2(-1)^n e^{-2\|\alpha\|^2} L_n(4\|\alpha\|^2)/\pi$
	$Q(\alpha)$	$\|\alpha\|^{2n} e^{-\|\alpha\|^2}/(\pi n!)$
Chaotic state $\hat{\rho} = (1 - e^{-\beta})$ $\sum_{n=0}^{\infty}\|n\rangle\langle n\|e^{-n\beta}$	$P(\alpha)$	$\frac{1}{\pi\langle N\rangle}\exp[-\|\alpha\|^2/\langle N\rangle]$
	$W(\alpha)$	$\frac{1}{\pi(\langle N\rangle+1/2)}\exp[-\frac{\|\alpha\|^2}{\langle N\rangle+1/2}]$ $= \frac{1}{\langle N\rangle+1}\sum_{n=0}^{\infty}W_n(\alpha)(\frac{\langle N\rangle}{\langle N\rangle+1})^n$
	$Q(\alpha)$	$\frac{1}{\pi(\langle N\rangle+1)}\exp[-\|\alpha\|^2/(\langle N\rangle + 1)]$
"Schrödinger's cat" state $\frac{\|a+ib\rangle+\|a-ib\rangle}{\{2[1+\cos(2ab)\exp(-2b^2)]\}^{1/2}}$	$P(\alpha)$	—
	$W(\alpha)$	$\frac{\exp[-2(\alpha_r-a)^2]}{\pi[1+\cos(2ab)\exp(-2b^2)]} * \{\exp[-2(\alpha_i - b)^2]$ $+ \exp[-2(\alpha_i + b)^2] + 2\exp[-2\alpha_i^2]\cos(4b\alpha_r - 2ab)\}$
	$Q(\alpha)$	$\frac{\exp[-(\alpha_r-a)^2]}{2\pi[1+\cos(2ab)\exp(-2b^2)]} * \{\exp[-(\alpha_i - b)^2]$ $+ \exp[-(\alpha_i + b)^2] + 2\exp[-b^2 - \alpha_i^2]\cos(2b\alpha_r)\}$
Even "Schrödinger's cat" state $\|\beta\rangle_e$	$P(\alpha)$	—
	$W(\alpha)$	$[e^{-2\|\beta-\alpha\|^2} + e^{-2\|\beta+\alpha\|^2} + 2e^{-2\|\alpha\|^2}\cos 4(\alpha_r\beta_i - \alpha_i\beta_r)]$ $/(\pi(1 + e^{-2\|\beta\|^2}))$
	$Q(\alpha)$	$\exp(-\|\alpha\|^2)\|e^{\beta\alpha^*} + e^{-\beta\alpha^*}\|^2/(4\pi\cosh(\|\beta\|^2))$
Odd "Schrödinger's cat" state $\|\beta\rangle_o$	$P(\alpha)$	—
	$W(\alpha)$	$[e^{-2\|\beta-\alpha\|^2} + e^{-2\|\beta+\alpha\|^2} - 2e^{-2\|\alpha\|^2}\cos 4(\alpha_r\beta_i - \alpha_i\beta_r)]$ $/(\pi(1 - e^{-2\|\beta\|^2}))$
	$Q(\alpha)$	$\exp(-\|\alpha\|^2)\|e^{\beta\alpha^*} - e^{-\beta\alpha^*}\|^2/(4\pi\sinh(\|\beta\|^2))$

Table 2.1: The P, Wigner and Q functions of some common quantum and classical states in quantum optics. (Those P functions that are not provided in this table don't exist.)

(a)

(b)

Figure 2.2: The Wigner functions of a vacuum state (a) and a squeezed vacuum state with 6 dB phase quadrature squeezing (b). They are both Gaussian distributions with non-negative value. The maximum possible value of these Wigner functions is $2/\pi$.

(a)

(b)

Figure 2.3: The Wigner functions of the Fock state $|2\rangle$ (a) and the even "Schrödinger's cat" state defined in Eq. (2.9) with $\alpha = 2$ (b). They are both non-classical distributions with negative value.

Chapter 3

Quantum optical measurements

3.1 Introduction

In this chapter we want to compare different quantum-optical measurements, i. e., direct intensity measurement, standard homodyne detection and optical homodyne tomography (OHT). Based on this comparison, we point out the ability and importance of OHT. We will also present a semiclassical theory to model the OHT system, which will be useful for us to explain some experimental results in chapter 6-8.

The focus of this chapter will be OHT and the inverse Radon transform, the latter is the essential mathematical tool for tomographic method. Tomography is a mature technology in many areas of science and technology. For example, the CAT-scan (computer assisted tomography) is widely used in medical science in order to diagnose the illness of human organs. In 1979, the Nobel prize in physiology or medicine was awarded to Allan M. Cormack and Godfrey N. Hounsfield for their contributions toward the development of computer assisted tomography, a revolutionary radiological method. Shown in Fig. 3.1 is an illustration of tomography.

Figure 3.1: Tomography, a technology widely used in medical science (i.e., the CAT scan). Two dimensional objects (the shaded area) can be reconstructed from their projections. Projections are integrals along certain directions.

We will also present a simplification of the inverse Radon transform in section 3.5 using an integral method. This may be useful in the numerical implementation of tomographic reconstruction.

3.2 Direct intensity measurement

A realistic light beam always has some amount of intensity and frequency noise. These are fluctuations of its intensity and frequency. Shown in Fig. 3.2 is a schematic diagram of direct intensity measurement of a light beam. This is the simplest optical measurement. The DC component of the photodetector output is monitored by an oscilloscope or a multimeter. This output voltage is proportional to the optical power of the light, which can also be measured by a power meter. The AC component is measured by a spectrum analyser (SA). This is a measurement of the noise power at RF sidebands as well as deliberately imposed modulation sidebands, if any.

Figure 3.2: Schematic of direct intensity measurement. PD: photodetector; CRO: oscilloscope; SA: spectrum analyser.

Suppose the quantum efficiency of the photodiode is η. The detected photocurrent is proportional to the number of photons incident on the surface of the photodiode,

$$\hat{i} = \eta e \hat{N} = \eta e \hat{a}^\dagger \hat{a}, \tag{3.1}$$

where e is the charge of an electron. A spectrum analyser can measure the variance of the photocurrent $V_{\hat{i}}$ which is proportional to the variance of the photon number $V_{\hat{N}}$

$$V_{\hat{i}} = \eta^2 e^2 V_{\hat{N}}. \tag{3.2}$$

Direct intensity measurement is simple and efficient. However, it's a phase insensitive measurement and can only measure the intensity noise of the light. In order to carry out a phase sensitive measurement, we have to employ homodyne detection, which will be discussed in the next section.

3.3 Standard homodyne detection

The schematic of homodyne detection [70–72] is shown in Fig. 3.3. A signal wave is mixed with a local oscillator wave at a 50:50 beam splitter. The signal and the local oscillator have a relative phase θ which is normally adjusted by controlling the path length of the local oscillator beam. The two output beams are detected by a pair of matched, high efficiency detectors PD1 and PD2, respectively. The difference current, $\hat{i}_-(t) = \hat{i}_1(t) - \hat{i}_2(t)$, is monitored with a spectrum analyser,

$$\hat{i}_-(t) = \hat{i}_1(t) - \hat{i}_2(t) = \eta e(\hat{d}^\dagger \hat{d} - \hat{c}^\dagger \hat{c}). \tag{3.3}$$

Figure 3.3: Schematic of homodyne detection. $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ are the annihilation operators of the corresponding ports. PD1, PD2: photodetectors; BS: beam splitter; SA: spectrum analyser.

If we use the annihilation operator relationship between the four ports,

$$\begin{aligned}
\hat{c} &= (\hat{a} - \hat{b})/\sqrt{2}, \\
\hat{d} &= (\hat{a} + \hat{b})/\sqrt{2},
\end{aligned} \tag{3.4}$$

it's easy to find that

$$\hat{i}_-(t) = \eta e(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}). \tag{3.5}$$

For a homodyne detection to be valid, we have to make the local oscillator wave much stronger than the signal wave[1], so that we can treat the local oscillator wave classically, the operator $\hat{b}$ is replaced by a complex number,

$$\hat{b} \rightarrow Be^{i\theta}. \tag{3.6}$$

Substituting Eq. (3.6) into Eq. (3.5) we find

$$\begin{aligned}
\hat{i}_-(t) &= \eta eB(\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta}) \\
&= 2\eta eB\hat{X}(\theta),
\end{aligned} \tag{3.7}$$

where $\hat{X}(\theta) = (\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta})/2$ is the rotated quadrature operator. Eq. (3.7) simply tells us that the measurement of the difference current $\hat{i}_-(t)$ is equivalent to the measurement

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[1]Normally the local oscillator intensity should be at least ten times that of the signal wave.

of the θ quadrature amplitude. If we change the local oscillator phase θ, we can measure any quadrature we want. This is a phase sensitive measurement.

What a spectrum analyser can measure is the noise equivalent power of the photocurrent $i_-(t)$, i. e., the variance $\Delta i_-{}^2(t)$. A spectrum analyser collects the time series $i_-(t)$, mixes it with an RF local oscillator, and detects the envelope of the mixer output. This is sufficient for variance measurement. However, it loses the information of photocurrent value itself. Variance is the second order moment of a probability density function (PDF). If higher order moments exist in the PDF, they can not be detected by a spectrum analyser.

3.4 Inverse Radon transform and optical homodyne tomography

Although homodyne detection is a phase sensitive measurement, it only gives the variance for each quadrature. This is sufficient for Gaussian states (see discussion of Gaussian states in section 2.5), because the average value and the variance can determine a Gaussian distribution completely. In general, measurements of variances are not sufficient to fully characterise quantum states. It is essential to measure all the quadrature amplitude distributions $w(x, \theta)$ if one wishes to preserve the complete information and extract the density matrix or the Wigner function of an arbitrary state. Here x is the quadrature amplitude value.

Tomography was first introduced into quantum mechanics by Bertrand and Bertrand [5]. They pointed out that the Wigner function can be reconstructed from marginals (projections). However, it's Vogel and Risken's pioneering paper [6] that brought a direct link between experiment and theory and inspired many works in this area of quantum optics. They pointed out that s-parameterised QPD's can be reconstructed from the measured PDF's $w(x, \theta)$,

$$W(\alpha_r, \alpha_i, s) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^\pi w(x, \theta) \exp[s\eta^2/8 + i\eta(x - \alpha_r \cos\theta - \alpha_i \sin\theta)]|\eta| dx d\eta d\theta.$$
(3.8)

The measurement of $w(x, \theta)$ is done by optical homodyning in the θ quadrature, and θ can be scanned by a PZT. η is an integrating parameter. Eq. (3.8) is a three-fold integration. It results from a complex inverse Fourier transform. The first experiment to reconstruct the Wigner function was carried out by Smithey, et al. [7–9]. The Wigner functions of a vacuum state and a quadrature-squeezed state were measured. This method was referred to as optical homodyne tomography (OHT) [7] because it's similar to the tomographic inversion from projections of two dimensional objects [73, 74]. Eq. (3.8) is called an inverse Radon transform [75].

3.5 Integral of the inverse Radon transform

Generally, for s-parameterised QPD, we have the inverse Radon transform Eq. (3.8). It's a three-fold integral. For the numerical implementation of Eq. (3.8), we have to calculate the Wigner function point by point. This is quite a time-consuming task. It would be nice if we can simplify Eq. (3.8) to save the computing time. For the Wigner function $W(\alpha_r, \alpha_i, s = 0)$, the variable η in Eq. (3.8) can not be integrated out. However, for $s < 0$, we can integrate out the parameter η.

By re-arranging the variables, we can re-write Eq. (3.8) as

$$W(\alpha_r, \alpha_i, s) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_0^\pi w(x, \theta) dx d\theta \left[\frac{16}{-s} \int_0^{+\infty} e^{-\eta^2} \eta \cos(\eta y) d\eta \right], \qquad (3.9)$$

where we have defined $y \equiv \sqrt{\frac{8}{-s}}(x - \alpha_r \cos\theta - \alpha_i \sin\theta)$. On the RHS we have such an integral [76]

$$\int_0^{+\infty} d\eta \eta e^{-\eta^2} \cos(\eta y) = \frac{1}{2}\phi(1, 1/2; -y^2/4) = \frac{1}{2} + \frac{1}{2} \sum_{n=0}^\infty \frac{1}{(2n+1)!!}(-y^2/2)^{n+1}, \quad (3.10)$$

where $\phi(\alpha, \gamma; z)$ is the degenerate hypergeometric function

$$\phi(\alpha, \gamma; z) = 1 + \frac{\alpha}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{z^3}{3!} + \ldots. \qquad (3.11)$$

Now we can substitute Eq. (3.10) into Eq. (3.9) and obtain a two-fold integral

$$\begin{aligned} W(\alpha_r, \alpha_i, s) &= -\frac{2}{\pi^2} \int_{-\infty}^{+\infty} dx \int_0^\pi d\theta w(x + \alpha_r \cos\theta + \alpha_i \sin\theta, \theta) \frac{1}{s}\phi(1, \frac{1}{2}; \frac{2x^2}{s}) \quad (3.12) \\ &= \frac{2}{\pi^2} \int_{-\infty}^{+\infty} dx \int_0^\pi d\theta w(\sqrt{-s}x + \alpha_r \cos\theta + \alpha_i \sin\theta, \theta) \frac{1}{\sqrt{-s}} \\ &\quad \phi(1, \frac{1}{2}; -2x^2). \end{aligned} \qquad (3.13)$$

For the Q function ($s = -1$),

$$\begin{aligned} Q(\alpha) &= W(\alpha, -1) \\ &= \frac{2}{\pi^2} \int_{-\infty}^{+\infty} dx \int_0^\pi d\theta w(x + \alpha_r \cos\theta + \alpha_i \sin\theta, \theta)\phi(1, \frac{1}{2}; -2x^2). \end{aligned} \qquad (3.14)$$

We note that Eq. (3.12) and Eq. (3.13) only hold for $s < 0$. There is no convergent integral in the case of the Wigner function ($s = 0$). However, we can always choose $s \to 0^-$ so that s is close enough to 0, and $W(\alpha, s)$ is close enough to the Wigner function $W(\alpha, 0)$. For a practical reconstruction algorithm, Eq. (3.12) is easier to handle because we can always choose the range of x to be an area in the phase space where $W(\alpha, s)$ is of significance (i.e., not too close to zero). For this range of x, we can calculate the function $\phi(1, 1/2; 2x^2/s)$ beforehand. When the data for the PDF's $w(x, \theta)$ is ready, we only need to do a two-fold integration as shown in Eq. (3.12) for the reconstruction of the Wigner function. This can save time in the algorithm. We note that another way of simplification was developed by Janicke and Wilkens [77]. Their method used the Dawson integral which can be numerically evaluated [78]. Both their method and ours presented above should have comparable effect for the reconstruction, because they all simplify the three-fold integral to a two-fold one.

3.6   Semiclassical theory of optical homodyne tomography

In this section we will model the optical homodyne tomography system with a semiclassical theory. We will include phase modulation in our model, and show how this signal can be retrieved in phase space together with the quantum noise.

3.6.1 Modeling of the OHT system

Figure 3.4: Experimental setup of OHT with a general demodulation scheme. EOM: electro-optic modulator; ND: neutral-density filter; PZT: piezo-electric transducer; PD: photodetector; LPF: low-pass filter; SA: spectrum analyser. There are two modes of operation. For a synchronous mode, the signal generator with frequency Ω_d is not used; while for an asynchronous mode, the signal with frequency Ω_m is not applied to the mixer, but the signal with frequency $\Omega_d (\neq \Omega_m)$ is.

In practical applications, such as sensing or communication, a laser beam is modulated in order to carry information. This can occur at different modulation frequencies or channels Ω_m. Each modulation frequency has its own Wigner function $W_{\Omega m}(x_1, x_2)$, and can vary dramatically from channel to channel [15, 79]. Hence, a realistic laser beam with signal modulations is a propagating multi-mode quantum state, which is quite different to the single mode intra-cavity description of an optical state. Such a realistic light beam can only be described by a spectrum of Wigner functions W_Ω, one for each frequency Ω. Experimentally we can measure the time traces of different quadrature amplitudes $\hat{X}_{\Omega d}(\theta)$ by scanning quadrature phase θ from 0 to π. An experimental setup of OHT with a general demodulation scheme is shown in Fig. 3.4. Please refer to section 7.3.1 for a detailed explanation of the setup. After binning these traces we get $w_{\Omega d}(x, \theta)$, the PDF's of different quadrature amplitudes at one detection frequency. The corresponding Wigner

function can be reconstructed from these PDF's [6, 32] using

$$W_{\Omega d}(x_1, x_2) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^\pi w_{\Omega d}(x, \theta) \exp[i\eta(x - x_1 \cos(\theta) - x_2 \sin(\theta))]$$
$$|\eta| dx d\eta d\theta, \tag{3.15}$$

which is the inverse Radon transform for $w_{\Omega d}(x, \theta)$. Eq. (3.15) is obtained from Eq. (3.8) by setting s to zero. Here (x_1, x_2) denotes a point in the complex phase space, θ is the quadrature angle and x is the quadrature amplitude value. η is a parameter that originates from an inverse Fourier transform (see Eq. (2.25)). From section 3.5 we know that Eq. (3.15) is in a critical point of convergence. Oscillation can easily occur. In order to smooth out the high spatial frequency oscillation in the transform, we sometimes also use a filtered version

$$W_{\Omega d}(x_1, x_2) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^\pi w_{\Omega d}(x, \theta) \exp[i\eta(x - x_1 \cos(\theta) - x_2 \sin(\theta))]|\eta|$$
$$\exp[-\gamma\eta^2] dx d\eta d\theta. \tag{3.16}$$

For roll-off parameter $\gamma \to 0$, Eq. (3.16) is the same as Eq. (3.15). The purpose of adding a Gaussian filtering factor $\exp[-\gamma\eta^2]$ is to filter out the high spatial frequency noise [74]. γ is always chosen to be very small so that the reconstructed Wigner function is precise enough to represent the original one.

Now we will model the OHT detection system using a semiclassical theory. In this model we include phase modulation, and show how it will represent itself in phase space. The traditional picture of a single Wigner function of a laser is shown to be not sufficient in an experimental case. The reason is that we can only reconstruct the Wigner function at RF frequencies, which is the sideband of the carrier frequency, while the strong coherent component of a laser is at the carrier frequency ($\Omega = 0$). A similar result was reported in [12], where only synchronous demodulation was used. However, in order to explain our experimental results, we have to consider a more general demodulation scheme. This is achieved by using different frequencies for modulation and demodulation respectively.

The output of phase modulator is given by

$$E_{out} = E_0 \cos[\omega t + \delta f(t)], \tag{3.17}$$

where δ is the modulation index, and

$$f(t) = \sin(\Omega_m t) \tag{3.18}$$

is the modulation signal. Phase modulation of optical field generates sideband pairs at $\omega \pm n\Omega_m$. We can rewrite the phase modulated signal wave as a decomposition into harmonics of the modulation frequency Ω_m. The first order approximation, assuming $\delta \ll 1$, gives

$$E_{out} = E_0 \cos(\omega t) + \beta E_0[\cos(\omega + \Omega_m)t - \cos(\omega - \Omega_m)t]$$
$$= \frac{E_0}{2}[e^{-i\omega t}(1 + \beta e^{-i\Omega_m t} - \beta e^{i\Omega_m t}) + e^{i\omega t}(1 + \beta e^{i\Omega_m t} - \beta e^{-i\Omega_m t})], \tag{3.19}$$

where $\beta = J_1(\delta)$ is the first order Bessel function. Using the correspondence principle,

we can assume

$$\hat{a}(t) \leftrightarrow E_0(1 + \beta e^{-i\Omega_m t} - \beta e^{i\Omega_m t}) \tag{3.20}$$

as the annihilation operator of the field (in the rotating frame). In order to include the quantum fluctuation, we express $\hat{a}(t)$ as

$$\begin{aligned}
\hat{a}(t) &= [E_0 + \delta\hat{a}(t)](1 + \beta e^{-i\Omega_m t} - \beta e^{i\Omega_m t}) \\
&\approx E_0 + \beta E_0(e^{-i\Omega_m t} - e^{i\Omega_m t}) + \delta\hat{a}(t) \\
&= E_0 - 2i\beta E_0 \sin(\Omega_m t) + \delta\hat{a}(t).
\end{aligned} \tag{3.21}$$

We assume the term $\delta\hat{a}(t)\beta(e^{-i\Omega_m t} - e^{i\Omega_m t})$ is much smaller and omit it. This is true when $|\beta| \ll 1$ (weak modulation). Because $\delta\hat{a}(t)$ is still in the equation and we will not make any further approximation, any non-Gaussian feature of the quantum states will be preserved in our calculation. In this sense, we have not used the linearisation approximation [80–82]. We treat the strong local oscillator beam classically,

$$\hat{b}(t) \approx Be^{i\theta}, \tag{3.22}$$

where θ is the local oscillator phase relative to the signal field and is scanned by a PZT (see Fig. 3.4). For a balanced homodyne detection, the difference current can be expressed by the field operators,

$$\hat{i}_-(\theta, t) = \hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}. \tag{3.23}$$

We have omitted the proportional coefficient which depends on the efficiencies of the detectors, losses and the electronic gain. With the following definitions:

$$\begin{aligned}
\hat{a}(t) &\equiv \hat{X}(t) + i\hat{Y}(t), \\
\hat{X}(t) &= Re[\hat{a}(t)], \\
\hat{Y}(t) &= Im[\hat{a}(t)], \\
\hat{X}(\theta, t) &\equiv \hat{X}(t)\cos(\theta) + \hat{Y}(t)\sin(\theta), \\
\delta\hat{X}(\theta, t) &\equiv \delta\hat{X}(t)\cos(\theta) + \delta\hat{Y}(t)\sin(\theta),
\end{aligned} \tag{3.24}$$

we get

$$\hat{X}(\theta, t) = E_0\cos(\theta) - 2\beta E_0 \sin(\Omega_m t)\sin(\theta) + \delta\hat{X}(\theta, t), \tag{3.25}$$

$$i_-(\theta, t) = 2B[E_0\cos(\theta) - 2\beta E_0 \sin(\Omega_m t)\sin(\theta) + \delta\hat{X}(\theta, t)]. \tag{3.26}$$

$\delta\hat{X}(\theta, t)$ describes the quantum fluctuation in the θ quadrature. In conventional experiments, the analysis of a particular Fourier component of the photocurrent is done by an RF spectrum analyser with the phase θ of the local oscillator kept constant. As a consequence the phase variance $V_{\Omega m}(\theta = \pi/2)$ increases proportional to the modulation depth while all other parts of the spectrum $V_\Omega(\theta = \pi/2)$ with $\Omega \neq \Omega_m$ remain unchanged. The local oscillator current of the mixer is a sinusoidal signal

$$i_l(t) \propto \sin(\Omega_d t + \psi), \tag{3.27}$$

where ψ is the electronic phase delay angle. Now we have the output current of the mixer

$$i_M(\theta, t) = i_-(\theta, t)i_l(t)$$

$$\begin{aligned}
\propto \quad & \sin(\Omega_d t)\cos(\psi)E_0\cos(\theta) - \sin(\Omega_d t)\cos(\psi)2\beta E_0\sin(\Omega_m t)\sin(\theta) \\
& + \sin(\Omega_d t)\cos(\psi)\delta\hat{X}(\theta,t) + \cos(\Omega_d t)\sin(\psi)E_0\cos(\theta) \\
& - \cos(\Omega_d t)\sin(\psi)2\beta E_0\sin(\Omega_m t)\sin(\theta) + \cos(\Omega_d t)\sin(\psi)\delta\hat{X}(\theta,t). \quad (3.28)
\end{aligned}$$

We define the Fourier transforms

$$\begin{aligned}
\tilde{i}_M(\theta,\Omega') &= F[i_M(\theta,t)] = \int_{-\infty}^{+\infty} i_M(\theta,t)e^{-i\Omega't}dt, \\
\delta\tilde{X}(\Omega') &= F[\delta\hat{X}(t)], \\
\delta\tilde{Y}(\Omega') &= F[\delta\hat{Y}(t)], \\
\delta\tilde{X}(\theta,\Omega') &= F[\delta\hat{X}(\theta,t)], \quad (3.29)
\end{aligned}$$

and obtain

$$\delta\tilde{X}(\theta,\Omega') = \delta\tilde{X}(\Omega')\cos(\theta) + \delta\tilde{Y}(\Omega')\sin(\theta). \quad (3.30)$$

Now the output current of the low-pass filter can be expressed as

$$\begin{aligned}
i_{\Omega d}(\theta,t) &= LPF[i_M(\theta,t)] = \frac{1}{2\pi}\int_{-\Delta\Omega_f}^{\Delta\Omega_f} \tilde{i}_M(\theta,\Omega')e^{i\Omega't}d\Omega' \\
&= -\beta E_0\sin(\theta)\delta(\Omega_d,\Omega_m) \\
&\quad + \frac{i}{2}\cos(\psi)\frac{1}{2\pi}\int_{-\Delta\Omega_f}^{\Delta\Omega_f}[\delta\tilde{X}(\theta,\Omega'+\Omega_d) - \delta\tilde{X}(\theta,\Omega'-\Omega_d)]e^{i\Omega't}d\Omega' \\
&\quad + \frac{1}{2}\sin(\psi)\frac{1}{2\pi}\int_{-\Delta\Omega_f}^{\Delta\Omega_f}[\delta\tilde{X}(\theta,\Omega'-\Omega_d) + \delta\tilde{X}(\theta,\Omega'+\Omega_d)]e^{i\Omega't}d\Omega', \quad (3.31)
\end{aligned}$$

where $\Delta\Omega_f$ is the bandwidth of the low-pass filter. It is about 50 kHz in our experiment[2]. Here we have defined a function $\delta(\Omega_d,\Omega_m)$,

$$\delta(\Omega_d,\Omega_m) = \cos[\psi + (\Omega_d - \Omega_m)t] \quad (3.32)$$

when $|\Omega_d - \Omega_m| < \Delta\Omega_f$ and

$$\delta(\Omega_d,\Omega_m) = 0 \quad (3.33)$$

when $|\Omega_d - \Omega_m| > \Delta\Omega_f$. If we define these finite-bandwidth time domain functions

$$\begin{aligned}
\delta\hat{X}_c(\Omega_d;t) &\equiv \frac{1}{2\pi}\int_{-\Delta\Omega_f}^{\Delta\Omega_f} \delta\tilde{X}(\Omega'+\Omega_d)e^{i\Omega't}d\Omega', \\
\delta\hat{Y}_c(\Omega_d;t) &\equiv \frac{1}{2\pi}\int_{-\Delta\Omega_f}^{\Delta\Omega_f} \delta\tilde{Y}(\Omega'+\Omega_d)e^{i\Omega't}d\Omega', \\
\delta\hat{X}_c(\theta,\Omega_d;t) &\equiv \delta\hat{X}_c(\Omega_d;t)\cos(\theta) + \delta\hat{Y}_c(\Omega_d;t)\sin(\theta), \quad (3.34)
\end{aligned}$$

we find

$$\begin{aligned}
\delta\hat{X}_c^\dagger(\Omega_d;t) &= \delta\hat{X}_c(-\Omega_d;t), \\
\delta\hat{Y}_c^\dagger(\Omega_d;t) &= \delta\hat{Y}_c(-\Omega_d;t), \\
\delta\hat{X}_c^\dagger(\theta,\Omega_d;t) &= \delta\hat{X}_c(\theta,-\Omega_d;t). \quad (3.35)
\end{aligned}$$

---

[2]The low-pass filter is used to select a slice of the noise spectrum. To study the single mode Ω_d, we have to make sure that $\Delta\Omega_f \ll \Omega_d$.

Thus we have two Hermitian operators

$$\delta \hat{X}_{cr}(\theta, \Omega_d; t) = (\delta \hat{X}_c(\theta, \Omega_d; t) + \delta \hat{X}_c(\theta, -\Omega_d; t))/2,$$
$$\delta \hat{X}_{ci}(\theta, \Omega_d; t) = (\delta \hat{X}_c(\theta, \Omega_d; t) - \delta \hat{X}_c(\theta, -\Omega_d; t))/2i. \quad (3.36)$$

We can roughly think of these operators as the real $(\delta \hat{X}_{cr}(\theta, \Omega_d; t))$ and imaginary $(\delta \hat{X}_{ci}(\theta, \Omega_d; t))$ parts of $\delta \hat{X}_c(\theta, \Omega_d; t)$. Here $\delta \hat{X}_c(\theta, \Omega_d; t)$ can be understood as the total quantum fluctuation centered around $\pm\Omega_d$. Now we can substitute these finite-bandwidth time domain functions into Eq. (3.31) to give

$$i_{\Omega d}(\theta, t) = -\beta E_0 \delta(\Omega_d, \Omega_m) \sin(\theta) - \cos(\psi) \delta \hat{X}_{ci}(\theta, \Omega_d; t) + \sin(\psi) \delta \hat{X}_{cr}(\theta, \Omega_d; t). \quad (3.37)$$

$i_{\Omega_d}(\theta, t)$ is the final electronic signal which is measured in the experiment. For synchronous detection, $\Omega_d = \Omega_m$ and $\delta(\Omega_d, \Omega_m) = \cos(\psi)$. Now if we further set $\psi = 0$ in the experiment

$$i_{\Omega_d}(\theta, t) = -\beta E_0 \sin(\theta) - \delta \hat{X}_{ci}(\theta, \Omega_d; t). \quad (3.38)$$

In this case $i_l(t) = \sin(\Omega_d t)$ is in phase with the modulation signal of Eq. (3.18).

The first term in Eq. (3.38) is due to the phase modulation. In phase space, it gives us a displacement from the origin (coherent amplitude, see, e. g., Ref. [83]). When $\psi = \pi/2$, $i_l(t) = \cos(\Omega_d t)$ is out of phase with the modulation signal, and $i_{\Omega_d}(\theta, t) = \delta X_{cr}(\theta, \Omega_d; t)$. The displacement frequently drawn in phase space diagrams no longer exists. Only $\delta X_{cr}(\theta, \Omega_d; t)$ is the signal measured and used as the quadrature amplitude value at the detection frequency chosen.

In order to visualise the different angles discussed in this modeling, we show in Fig. 3.5 an illustration of the three angles θ (optical local oscillator quadrature angle), ψ (electronic local oscillator phase delay angle) and φ (quadrature phase angle).

3.6.2 Reconstruction procedures

In order to obtain the Wigner function of a light beam, only small modifications to the balanced homodyne apparatus are required [11]. Phase synchronous detection is introduced by replacing the spectrum analyser with a mixer demodulator (see Fig. 3.4). The mixer is gated by an electronic local oscillator signal derived from the same generator that drives the EOM. This electronic signal is shifted by a phase ψ giving a mixed down difference current $i_{\Omega m}(\theta, \psi; t)$. Let $\Omega_d = \Omega_m$ in Eq. (3.37) we can write down the output current from the mixer as :

$$i_{\Omega m}(\theta, \psi; t) = -\beta E_0 \cos(\psi) \sin(\theta) - \cos(\psi) \delta X_{ci}(\theta, \Omega_m; t) + \sin(\psi) \delta X_{cr}(\theta, \Omega_m; t), \quad (3.39)$$

where $\delta X_c(\theta, \Omega_m; t)$ can be understood as the total quantum fluctuations centered around $\pm\Omega_m$. $\delta X_{ci}(\theta, \Omega_m; t)$ and $\delta X_{cr}(\theta, \Omega_m; t)$ are the imaginary and real parts of $\delta X_c(\theta, \Omega_m; t)$ respectively. For $\psi = 0$, we obtain

$$i_{\Omega m}(\theta, 0; t) = -\beta E_0 \sin(\theta) - \delta X_{ci}(\theta, \Omega_m; t). \quad (3.40)$$

The first term contains all the modulation and the second term all the quantum fluctuations. Note that for synchronous detection, the phase of the modulation has to be known to the observers. In practice, this is not always possible.

If the demodulation phase ψ is unavailable to the detection system, as would be the

Figure 3.5: Different angles in a quantum state measurement. θ: optical local oscillator quadrature angle; ψ: electronic local oscillator phase delay angle; φ: quadrature phase angle corresponding to maximum fluctuation. The length of the displacement is proportional to $\cos(\psi)$ for synchronous detection ($\Omega_d = \Omega_m$). In most experiments described in this thesis ψ is optimised, that means $\psi = 0$, in order to measure the modulation at Ω_m with maximum sensitivity.

case for the monitoring of remotely generated signals, then either a phase recovery technique or asynchronous demodulation is required. Asynchronous demodulation can be simulated by Eq. (3.39) where the demodulation phase ψ is a linear function of time. This corresponds to an uncorrelated demodulation generator operating at Ω_d as shown in Fig. 3.4.

A typical synchronously demodulated photocurrent plot is shown in Fig. 3.6, where the phase angle θ is repetitively scanned. By selecting data that correspond to the same value inside the vertical intervals $\delta\theta$, we obtain measurements of $w_{\Omega d}(\theta, t)$ for any given quadrature interval $(\theta, \theta + \delta\theta)$. Next a histogram of this current is formed by binning the data in intervals $\delta\theta$ for a coherent state. This results in the PDF $w_{\Omega d}(x, \theta)$ of the quadrature amplitude for various projection angle θ. Fig. 3.7 shows a series of such PDF's for a full scan of θ.

The width of a Gaussian PDF corresponds to the variance $V(\theta)$ of the given quadrature. For any state we are able to generate, and thus linearisable state with photon number $N \gg 1$, the shape of the PDF is a Gaussian. For a coherent state the width of the Gaussian is equal to the photon number, thus identical to a Poissonian distribution. Squeezed states have sub-Poissonian PDF's at one particular angle θ_s, the squeezing quadrature.

Figure 3.6: One segment of the measured time trace of quadrature amplitude value by scanning the PZT in a typical OHT experiment. 5 dBm of phase modulation at 7 MHz was applied to a laser beam. The detection bandwidth is 50 kHz. a.u.: arbitrary unit.

3.7 Summary

We summarised some quantum-optical measurements, i. e., direct intensity measurement, standard homodyne detection and OHT. From the comparison we see that it's essential to carry out OHT in order to completely characterise a quantum state. The inverse Radon transform is the mathematical tool of OHT. We provided an integral of the inverse Radon transform, and this may save computing time in a real reconstruction procedure of the Wigner function. A semiclassical theory was developed to model the OHT system. Phase modulation and asynchronous demodulation were included in this theory and the results in terms of detected photocurrent were discussed.

Figure 3.7: The probability density function (PDF) after binning the time trace. unit of the L. O. angle: radian.

Chapter 4

Data acquisition, binning and numerical methods

4.1 Introduction

Unlike the direct probing of phase space [26–28], optical homodyne tomography involves more sophisticated numerical methods. In order to get the quadrature distributions, we have to scan the local oscillator phase from 0 to π (see Section 3.6.2). The resulting long trace of data points must be binned into different quadratures and different quadrature values for each quadrature.

In this chapter we will first show the details of data acquisition and binning. Then we will present the numerical implementation of the inverse Radon transform. Simulation of quantum state reconstruction follows to verify the validity of this implementation. Numerical filtering will also be presented to deal with noisy data.

4.2 Data acquisition and binning

Shown in Fig. 3.6 is a segment of the raw data from OHT of a modulated coherent state. It consists of 1,000 data points. The whole time series consists of 200,000 data points. This trace is first binned into different quadratures. There is a second binning procedure for each quadrature in order to get the probability density functions (PDF's). The number of quadratures needed depends on the quantum state we want to measure. It's often chosen to be 120 in our experiments. Too few quadratures will result in poor resolution of the Wigner function. In the case of a squeezed vacuum state, this is a phase averaging effect which will artificially degrade the degree of squeezing. Too many quadratures is not necessary as long as we can have good enough resolution for the Wigner function. In practical cases, too many quadratures impose a heavier numerical task.

In order to bin the quadrature value trace into different quadratures, we have to simultaneously acquire another trace which contains the optical phase information. The first idea is to use the scanning voltage which drives the piezo-electric transducer (PZT). However, this is not reliable. Due to hysteresis in the response of the PZT, the scanning voltage does not give an accurate measurement of the optical phase. We show in Fig. 4.1 and Fig. 4.2 the probability density functions (PDF's) resulting from such phase reference. The traces corresponding to the increasing and decreasing stages of the scanning cross over each other. For larger modulation signal, this effect is more severe. The hysteresis of the PZT response is clearly displayed by the PDF's. The phase response is advanced in the increasing stage of the scanning while delayed in the decreasing stage. These PDF's can not be used for the reconstruction of the Wigner function. From this we also know that OHT is a very sensitive measurement technique.

In our experiments, an additional signal is acquired which allows the optical phase difference to be determined from the first-order interference of the beams in the arms

of the interferometer. This interference trace is a sine or cosine of the optical phase θ. We found that this trace is normally smooth enough for determining θ. However, with increased sampling rate, it can become rugged and we have to do a running average. The quadrature amplitude value trace together with the interference trace are sufficient for binning the experimental data into quadratures. There are some technical problems in the binning of data. In some sense these problems make binning more difficult than the implementation of the inverse Radon transform, which will be discussed in Section 4.3. Here we will discuss the most prominent problems.

Due to the DC offset of the operational amplifiers (op amp's), the acquired data has to be processed so that DC offset can be deducted. In the case of phase modulation, this is more difficult. First we need to determine the amount of modulation. This was done by calculating the average of the amplitude quadrature value and the phase quadrature value. After the modulation was determined, we deducted the original data from this amount of modulation. This is equivalent to shifting the displaced Wigner function back to the origin of phase space (see Section 3.6.2). We found it much easier to do it this way rather than do the inverse Radon transform directly for the displaced Wigner function. This is mainly a concern of grids needed for the algorithm. In doing so we can considerably reduce the computing time and improve the reconstruction accuracy. We call this the *center-seeking algorithm*. Because the correction of DC offset can not be perfect, a procedure is normally taken to adjust this minor imperfection. We calculate the average value of each quadrature and shift it to zero. Finally, for systematic consideration, we have another procedure to normalise the PDF's so that the integral of each PDF is 1. This is performed just to make sure we have reasonable PDF's. However, for all our PDF's, the deviations of their integrals from 1 are well within 0.1 percent. For the C program of data binning please refer to Appendix A.1. We show in Fig. 4.4 the flow chart of this binning program.

4.3   Numerical implementation of the inverse Radon transform

In the actual reconstruction of the Wigner function, we use the discrete version of the inverse Radon transform[1]

$$
\begin{aligned}
W(p_i, q_j, 0) \;=\; & \frac{1}{2\pi^2} \sum_{k=-(N_x-1)}^{N_x-1} \sum_{l=0}^{N_\eta-1} \sum_{n=0}^{N_\theta-1} w(x_k, \theta_n)\eta_l \cos[\eta_l(x_k - p_i \cos(\theta_n) - q_j \sin(\theta_n))] \\
& \Delta\eta\Delta x\Delta\theta,
\end{aligned} \tag{4.1}
$$

where $w(x_k, \theta_n)$ are the PDF's. We have chosen the range $x \in [-x_m, x_m]$, $\eta \in [0, \eta_m]$ so that the divisions $\Delta x = x_m/(N_x - 1)$ and $\Delta\eta = \eta_m/(N_\eta - 1)$. $2N_x - 1$ and N_η are the grid numbers of the quadrature values and the integrating parameter η respectively. x_m is determined by the range of measured quadrature amplitude values. We note that η originates from inverse Fourier transform. Δx and $\Delta\eta\Delta\theta$ form the pair of integration parameters with the same dimension. The choosing of η_m is tricky. Too small value of η_m results in poor resolution of the Wigner function. Too large value of η_m brings about oscillation effect. The value of η_m has to be adjusted in the reconstruction process. It has a filtering effect on the reconstructed Wigner function.

---

[1]Please refer to section 3.4 for the continuous version of the inverse Radon transform.

Please refer to Appendix A.2 for the C program of the inverse Radon transform. There are totally five loops of iteration in the reconstruction algorithm. In our program, some techniques are employed to reduce the computing time. For example, some common parameters are calculated beforehand and stored in arrays. We show in Fig. 4.5 and Fig. 4.6 the flow chart of the program for the inverse Radon transform.

4.4 Simulation of quantum state reconstruction

To verify the validity of our numerical implementation of the inverse Radon transform, we carry out some simulation of quantum state reconstruction. From quasiprobability distributions (QPD's) we can calculate the probability distributions of the rotated quadrature phase amplitude $\hat{x}(\theta)$ [6],

$$
\begin{aligned}
w(x,\theta) \;=\;\; & \frac{1}{2\pi}\int\int\int W\left(u\cos(\theta)-v\sin(\theta),u\sin(\theta)+v\cos(\theta),s\right) \\
& \exp[-s\eta^2/8+i(u-x)\eta]du\,dv\,d\eta.
\end{aligned}
\tag{4.2}
$$

The rotated quadrature phase amplitude $\hat{x}(\theta)$ is the linear combination of the creation and annihilation operators,

$$
\hat{x}(\theta)=(\hat{a}^\dagger e^{i\theta}+\hat{a}e^{-i\theta})/2.
\tag{4.3}
$$

In the case of the Wigner function, we have the discrete version of Eq. (4.2),

$$
\begin{aligned}
w(x_i,\theta_j) \;=\;\; & \sum_{n=0}^{N_v-1}\Delta v[W\left(x_i\cos(\theta_j)-n\Delta v\sin(\theta_j),x_i\sin(\theta_j)+n\Delta v\cos(\theta_j),0\right)+ \\
& W\left(x_i\cos(\theta_j)+n\Delta v\sin(\theta_j),x_i\sin(\theta_j)-n\Delta v\cos(\theta_j),0\right)].
\end{aligned}
\tag{4.4}
$$

In this simulation, we use 60 quadratures and 81 bins of quadrature amplitude value for each quadrature. The quantum state to be simulated is the Fock state $|1\rangle$. It's a non-Gaussian state and has negative value for the Wigner function. The reconstructed Wigner function is shown in Fig. 4.7. It's almost the same as the ideal one. However, the experimental data can be noisy, and this will of course affect the reconstruction. In order to simulate experimental fluctuations, we estimate typically 5% of fluctuation and add this amount of fluctuation to the PDF's,

$$
w(x,\theta)\Rightarrow w'(x,\theta)=w(x,\theta)\times k,
\tag{4.5}
$$

where $w(x,\theta)$ are the precise PDF's and $w'(x,\theta)$ are the noisy PDF's. $0.95\le k<1.05$ is a random number. The result is shown in Fig. 4.8. We can see the resulting Wigner function is not as smooth as the one shown in Fig. 4.7. However, the shape and the negativity of the Wigner function are maintained. From this simulation we see that our numerical reconstruction is valid.

4.5 Numerical filtering

As discussed in Section 4.3, we limit η to a cutoff value η_m, rather than ∞. This is a filtering of the Wigner function as discussed in Ref. [32]. Because the integral of the inverse Radon transform of the Wigner function is at a critical point of convergence (see

section 3.5), it is not a stable one. We can easily get oscillating features in the numerical implementation. In our implementation, we sometimes use a Gaussian filter for some really noisy data. We add a Gaussian roll-off factor to the inverse Radon transform,

$$W(x_1, x_2) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^\pi w(x, \theta) \exp[i\eta(x - x_1\cos(\theta) - x_2\sin(\theta))]|\eta|$$
$$\exp[-\gamma\eta^2]dxd\eta d\theta, \tag{4.6}$$

where $\exp[-\gamma\eta^2]$ is the Gaussian filter. Comparing with Eq. (3.8) we note that this is equivalent to reconstructing the -8γ-ordered QPD's.

In practice, we always choose γ small enough so that the accuracy is maintained while the filtering effect is achieved. For an example of quantum state reconstruction utilising numerical filtering, please refer to section 7.3.5. Due to asynchronous detection, the data is relatively noisy in this example.

4.6   Accuracy, sensitivity and validity

The inverse Radon transform is very sensitive to noise in the PDF's. Any error in the PDF's will propagate through the inverse Radon transform and shows up in the resulting Wigner function. The most important thing is to get accurate data for the PDF's. In practical cases, this means higher sampling rate, more stable optical phase reference, higher sampling accuracy and lower electronic noise. For the classical state tomography, we used a sampling I/O board with a sampling rate of 20 kSa/s. For the quantum state tomography, we used an oscilloscope which has a sampling rate of 2 GSa/s. This is mainly a concern of the long term phase stability of the optical system. Negativity is a common problem in the inverse Radon transform. Although we know there should be no negative value for our Wigner functions (negativity is a feature of non-classical states), small amount of negativity will normally occur in our results. The absolute value of the minimum (which is negative) is typically below 5% of the maximum of the Wigner function. This shouldn't be confused with the non-classical states. It's simply due to the imperfection of the data and the oscillating feature of the integral.

The accuracy and validity of the reconstructed Wigner function mainly rely on the accuracy of the PDF's. The effort should be devoted to the making of accurate PDF's. Then we are sure the reconstructed Wigner function will be accurate. For example, we should properly set the gain of the sampling I/O board, and electronically amplify the photocurrent from the experiment, so that the acquired data is accurate.

4.7   Summary

We presented in this chapter the numerical aspect of quantum state reconstruction (QSR). This includes the data acquisition, binning and the inverse Radon transform. We addressed the problem of hysteresis in the response of a PZT. This problem was solved by extracting precise optical phase from the interference trace. C programs were developed to fulfill all the numerical tasks involved in QSR. We also discussed noise and numerical filtering in the reconstruction algorithm. The validity of our numerical method was verified by a simulation of QSR of the Fock state $|1\rangle$, a highly non-Gaussian quantum state.

(a)

(b)

Figure 4.1: Probability density function (PDF) (a) and its contour plot (b) of a phase modulated coherent state (10 dBm RF driving). Scanning voltage of the PZT (from -0.34 V to 0.26 V) was used as phase reference. The left branch of the PDF corresponds to the decreasing stage of the scanning and the right branch corresponds to the increasing stage of the scanning. 200,000 data points were acquired. The quadrature amplitude is normalised by quantum noise limit; unit of the L. O. angle: radian.

(a)

(b)

Figure 4.2: Probability density function (PDF) (a) and its contour plot (b) of a phase modulated coherent state (15 dBm RF driving). Scanning voltage of the PZT (from -0.34 V to 0.26 V) was used as phase reference. The left branch of the PDF (between $2\pi/3$ and $4\pi/3$) corresponds to the decreasing stage of the scanning and the right branch corresponds to the increasing stage of the scanning. 200,000 data points were acquired. Because of larger modulation (comparing with the one in Fig. 4.1), the crossing of the two stages is obvious. The quadrature amplitude is normalised by quantum noise limit; unit of the L. O. angle: radian.

Figure 4.3: Center-seeking algorithm. Originally the Wigner function is evaluated in the (x_1, x_2) coordinate system and has a displacement from the origin. We deduct this amount of displacement from the data and effectively the Wigner function is shifted to the (x'_1, x'_2) coordinate system where there is no displacement for the Wigner function. Numerical task of reconstruction is considerably reduced by this algorithm.

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                  │ Definition of variables and arrays │
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                  │ Read control parameter data file │
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           │ Calculate the minimum and maximum of the interference trace │
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                  │ Deduct quadrature amplitude from the │
                  │ estimated DC offset and do multiplication │
                  └──────────────────────────────┘
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                  │ Calculate total countings for each quadrature │
                  └──────────────────────────────┘
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           │ Calculate quadrature amplitude probability density functions │
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                  └──────────────────────────────┘
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                                ▼
                           ┌─────────┐
                           │   End   │
                           └─────────┘
```

Figure 4.4: Flow chart of the binning program.

```
┌─────────┐
│  Start  │
└─────────┘
     │
     ▼
┌──────────────────────────────┐
│ Definition of variables and arrays │
└──────────────────────────────┘
     │
     ▼
┌──────────────────────────────┐
│ Read control parameter data file │
└──────────────────────────────┘
     │
     ▼
┌──────────────────────────┐
│ Check control parameters │
└──────────────────────────┘
     │
     ▼
┌──────────────────────────────┐
│ Calculate divisions for parameters │
└──────────────────────────────┘
     │
     ▼
┌────────────────────────────────────────────────┐
│ Prepare quadrature amplitude and phase space position arrays │
└────────────────────────────────────────────────┘
     │
     ▼
┌──────────────────────────────────────────────────────────┐
│ Read in quadrature amplitude probability density function (PDF) data file │
└──────────────────────────────────────────────────────────┘
     │
     ▼
┌──────────────────────────────────────────┐
│ Dump bad quadratures on the edges of the scanning │
└──────────────────────────────────────────┘
     │
     ▼
┌──────────────────────────────────────┐
│ Write new quadrature amplitude PDF data file │
└──────────────────────────────────────┘
     │
     ▼
┌──────────────────────────────────────────┐
│ Calculate DC offset of the quadrature amplitude data │
└──────────────────────────────────────────┘
     │
     ▼
┌──────────────────────────────────────────────────┐
│ Remove DC offset from the quadrature amplitude PDF and │
│    calculate the displacement of the state in phase space │
└──────────────────────────────────────────────────┘
     │
     ▼
┌──────────────────────────┐
│ Write data file for parameters │
└──────────────────────────┘
     │
     ▼
     ■ ■ ■
```

Figure 4.5: Flow chart of the program for the inverse Radon transform (part I).

Figure 4.6: Flow chart of the program for the inverse Radon transform (part II).

Figure 4.7: The reconstructed Wigner function of the Fock state $|1\rangle$ (numerical simulation). There is no added noise.

Figure 4.8: The reconstructed Wigner function of the Fock state $|1\rangle$ (numerical simulation). $5\ \%$ of random fluctuation is added to the PDF's (please see the text part in section 4.4).

Chapter 5

Optics and electronics for optical homodyne tomography

5.1 Overview

There are some commonly used optics and electronics in a typical optical homodyne tomography (OHT) experiment. In this chapter we want to summarise the use of these elements. We first introduce the photodetector, which is customised for our experiments. Then we outline the technical details of the diode pumped Nd:YAG laser. The third element we want to explain is the optical cavity. High finesse optical cavities were used in our laser noise measurement experiment, which will be discussed in Chapter 6. Finally we present the electronic configuration of OHT system. The use of an I/O board will be explained. Some technical problems and solutions in the data acquisition will be discussed.

5.2 Photodetector

A very important element in our experiment is photodetector. We used Epitaxx ETX500 InGaAs photodiodes for the detection of radiation at 1064 nm. The obtainable quantum efficiency of these photodiodes can be as high as 94%. Customised low noise amplification circuits were built into the photodetectors. (These circuits were designed by M. B. Gray [84]. Please refer to Fig. C.1 for the circuit diagram.) Two stages of operational amplifiers were used. The front end stage is a transimpedance operational amplifier (op-amp) uses a Comlinear CLC420 op-amp. The second stage provides a gain of +3.5 using a CLC430 in a non-inverting op-amp circuit. The overall 3 dB bandwidth of our photodetector circuit is 20 MHz. The noise equivalent power (NEP) is estimated to be 6 pW/$\sqrt{\text{Hz}}$ [84]. There is also arrangement for isolating the 50 Hz power supply signal in the design of our photodetectors. This is necessary for the sensitive low noise detection of light in our experiment. We show in Fig. 5.1 a trace of the laser intensity noise directly measured by one of our photodetectors and recorded by a spectrum analyser (SA).

5.3 Diode pumped Nd:YAG laser

All the lasers used in our experiments are diode pumped Neodymium doped Yttrium Aluminum Garnet (Nd:YAG) lasers. The laser crystal is a monolithic non-planar ring oscillator (NPRO) [85, 86]. The pump laser diode emits in a single discrete output at 808 nm. It can be temperature tuned to a strong absorption band of the gain media. The output wavelength of the Nd:YAG gain media is 1064 nm. The frequency tuning of the diode pumped Nd:YAG laser can be achieved by two ways. Temperature tuning is relatively slow with a time constant of approximately 1-10 seconds. The tuning range can

Figure 5.1: Laser intensity noise directly measured by a photodetector (Epitaxx ETX500 InGaAs) and recorded by a spectrum analyser. The intensity fluctuation of a laser beam is expanded into an RF spectrum, and the amplitude of this spectrum is called noise power. The frequency for signal roll-off is beyond 30 MHz for this particular photodetector. power: 2 mW; wavelength: 1064 nm; RBW: 100 kHz; VBW: 3 kHz. (The dip at 17.5 MHz is due to the amplification circuit built into the photodetector.)

be tens of GHz. Fast tuning is achieved by applying a voltage to a piezoelectric element bonded onto the crystal. This tuning range can be tens of MHz at modulation rates up to 100 kHz. This piezoelectric element is often used for frequency locking purpose in our experiment.

Three different lasers were used in our experiments. They are Lightwave 120, Lightwave 122 from Lightwave Electronics Inc. and Mephisto 700 from Laser Zentrum Hannover/InnoLight Inc. The output powers are 40 mW, 200 mW and 700 mW respectively.

5.4   Optical cavity

Three high finesse cavities were constructed in our experiments. In general they can be used as mode cleaners. The transmitted beam of a high finesse optical cavity has improved high RF noise performance (above the cavity linewidth). Comparing with the elliptic spatial mode of a diode pumped Nd:YAG laser, the spatial profile of the transmitted beam of an optical cavity is also very much improved because it's the transmitted TEM00 mode. In our laser phase noise measurement experiment, which will be discussed in chapter 6, a high finesse cavity was deliberately used to separate the high RF noise of the laser from the transmitted beam of the cavity[1].

Now let's discuss the calculation of a general optical cavity. (Note that we only discuss the on-resonance situation.) In Fig. 5.2 we show a ring cavity.

We denote T_i, A_i and R_i as the power transmittance, loss and reflectance of the mirrors. A_i can be the absorption and scattering of the mirrors. Suppose M1 is the input

---

[1]In this case the cavity was not used as a spatial mode cleaner.

Figure 5.2: A ring cavity. M1, M2 and M3 are mirrors. T_i, A_i and R_i denote the power transmittance, loss and reflectance of the mirrors. P_i: incident power; P_r: reflected power; P_t: transmitted power.

coupler and M2 is the output coupler. E_i, E_r and E_t are the incident, reflected and transmitted amplitude of the fields while P_i, P_r and P_t are the corresponding powers. For each mirror,

$$T_i + A_i + R_i = 1, \tag{5.1}$$

and

$$\begin{aligned} t_i &= \sqrt{T_i}, \\ a_i &= \sqrt{A_i}, \\ r_i &= \sqrt{R_i}. \end{aligned} \tag{5.2}$$

are the amplitude transmittance, loss and reflectance of the mirrors respectively. We have these formulas for the transmitted and reflected amplitudes [87],

$$\frac{E_t}{E_i} = -\frac{t_1 t_2}{\sqrt{r_1 r_2}} \frac{\sqrt{g}}{1 - g}, \tag{5.3}$$

$$\frac{E_r}{E_i} = r_1 - \frac{t_1^2}{r_1} \frac{g}{1 - g}, \tag{5.4}$$

where $g = r_1 r_2 \sqrt{1 - L_i}$ is the round-trip amplitude gain and $L_i = A_3 + T_3$ is the intra-cavity loss. For the transmitted and reflected powers we have

$$\frac{P_t}{P_i} \approx \frac{4 T_1 T_2}{(T_1 + T_2 + L)^2}, \tag{5.5}$$

$$\frac{P_r}{P_i} \approx \left(\frac{T_2 - T_1 + L}{T_1 + T_2 + L}\right)^2, \tag{5.6}$$

where $L = A_1 + A_2 + L_i$ is the total cavity loss. Here in Eq. (5.5) and (5.6) we have assumed very good mirrors, i.e., $A_i \ll 1, T_i \ll 1$. The finesse of the cavity is given by

$$\mathcal{F} = \frac{\pi\sqrt{g}}{1-g}. \tag{5.7}$$

For the circulating field amplitude inside the cavity we have

$$\frac{E_{circ}}{E_i} = \frac{jt_1}{1-g}. \tag{5.8}$$

The circulating power can be expressed by the Finesse,

$$\frac{P_{circ}}{P_i} = \frac{\mathcal{F}^2 T_1}{\pi^2}. \tag{5.9}$$

The free spectral range (FSR) is determined by the cavity optical length,

$$\mathrm{FSR} = \frac{c}{np}, \tag{5.10}$$

where p is the round-trip length of the cavity and n is the refractive index of the intra-cavity media (for air, $n = 1$). In the case of a linear cavity $p = 2l$ is the doubled cavity length. The full-width-half-maximum (FWHM) of a cavity is given by

$$\mathrm{FWHM} = \mathrm{FSR}/\mathcal{F}. \tag{5.11}$$

FWHM tells us the corner frequency of the cavity when used to transmit a "clean" mode. From Eq. (5.4) we can work out the condition for the reflected amplitude to be zero and the result is

$$R_1 = (1 - A_1)^2 R_2 (1 - L_i). \tag{5.12}$$

This is the *impedance matched* case. If the input coupler has no loss, i. e., $A_1 = 0$, Eq. (5.12) can be simplified as

$$R_1 = R_2 (1 - L_i). \tag{5.13}$$

When $R_1 > R_2(1 - L_i)$, the cavity is *under-coupled*; on the other hand, if $R_1 < R_2(1 - L_i)$, the cavity is *over-coupled*. The phase change experienced by the reflected fields are different in these two cases. For an under-coupled cavity, the phase moves up and down less than $\pi/2$ radians; for an over-coupled cavity, the phase of the reflected field shifts -180 degree when the cavity goes through resonance.

impedance matched:	$R_1 = R_2(1 - L_i)$
under-coupled:	$R_1 > R_2(1 - L_i)$
over-coupled:	$R_1 < R_2(1 - L_i)$

Table 5.1: Impedance matching of an optical cavity with lossless input coupler.

Now, in the case of a linear cavity (see Fig. 5.3), we provide a method for characterising the mirrors. For high reflectance mirrors, according to Eq. (5.5) and Eq. (5.6),

Figure 5.3: A linear cavity. M1 and M2 are mirrors. T_i, A_i and R_i denote the power transmittance, loss and reflectance of the mirrors. P_{r1}: reflected power; P_t: transmitted power.

$$\frac{P_{r1}}{Pi} \approx \left(\frac{T_2 - T_1 + A}{T_1 + T_2 + A}\right)^2, \tag{5.14}$$

$$\frac{P_{r2}}{Pi} \approx \left(\frac{T_1 - T_2 + A}{T_1 + T_2 + A}\right)^2, \tag{5.15}$$

$$\frac{P_{t1}}{Pi} \approx \left(\frac{4T_1 T_2}{T_1 + T_2 + A}\right)^2, \tag{5.16}$$

where P_i is the incident power, P_{r1} is the reflected power from M1 when M1 is the input coupler and P_{r2} is the reflected power from M2 when M2 is the input coupler. P_t is the transmitted power, which should be the same no matter which mirror (M1 or M2) is the input coupler. P_{r1}, P_{r2}, P_t, P_i can be easily measured in the experiment. Here we give an experimental example. For a linear cavity (cavity No. 1 in Table 5.2), we measured

$$\begin{aligned} P_{r1}/P_i &= 0.206, \\ P_{r2}/P_i &= 0.421, \\ P_t/P_i &= 0.514, \end{aligned} \tag{5.17}$$

and the Finesse of the cavity was measured to be

$$\mathcal{F} = 4,000. \tag{5.18}$$

Combining Eq. (5.14)-(5.18) we can readily obtain the parameters (ignoring unphysical solutions)

$$\begin{aligned} T_1 &= 0.11\%, \\ A_1 &= 77\ ppm, \\ R_1 &= 99.88\%, \\ T_2 &= 276\ ppm, \\ A_2 &= 77\ ppm, \\ R_2 &= 99.96\%, \end{aligned} \tag{5.19}$$

where we have assumed the same amount of loss for the two mirrors, i. e., $A_1 = A_2 = A/2^2$ (Note that 1 ppm=0.0001%).

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[2]In order to avoid contamination of the high quality mirrors as much as possible, we did not make sepa-

Listed in Table 5.2 are three high finesse cavities used in our experiments. Cavity No. 1

Cavity	No. 1	No. 2	No. 3
Type	Linear	Linear	Ring
Parameters	$T_1 = 0.11\%$	$T_{1,2} = 174\ ppm$	$T_{1,2} = 680\ ppm$
	$T_2 = 276\ ppm$		$R_3 = 99.999\%$
	$A_1 = 77\ ppm$	$A_{1,2} = 100\ ppm$	$A_{1,2} = 100\ ppm$
	$A_2 = 77\ ppm$		
	$R_1 = 99.88\%$	$R_{1,2} = 99.97\%$	$R_{1,2} = 99.92\%$
	$R_2 = 99.96\%$		
$p = 2l$	$34\ cm$	$32\ cm$	$42\ cm$
FSR	882 MHz	938 MHz	714 MHz
FWHM	220 kHz	67 kHz	179 kHz
$\mathcal{F}$	4,000	14,000	4,000

Table 5.2: High finesse cavities used in the experiments.

was used in the squeezed vacuum experiment which will be discussed in chapter 8. It was used as a mode cleaner for the local oscillator. Cavity No. 2 and No. 3 were used in the laser phase noise measurement experiment (see chapter 6). No. 2 has a very small FWHM and enabled us to measure the lower RF sidebands. No. 1 and No. 2 were constructed by SuperMirrors from Newport Inc.. These mirrors have ultra-low loss. No. 3 was constructed by low loss mirrors from Research Electro-Optics Inc. and can handle as high as 1 Watt of input optical power due to their excellent coatings [88].

5.5   Devices, elements and electronic configuration

The electronic configuration of OHT is shown in Fig. 5.4. It is basically an RF demodulation scheme. There are a few elements added in order to solve some technical problems.

There are two channels acquired by the I/O board, one is the quadrature amplitude and the other one is the optical phase. We acquire the DC component of either i_1 or i_2 as the optical phase. i_1 and i_2 are the photocurrents detected by a pair of photodetectors. Ideally the input impedance of the I/O board is large enough to prevent loading of the monitored photocurrent, say, i_2. However, there may be capacitive loading due to the cables. This will affect the balanced subtraction of i_1 and i_2. So we add a buffer amplifier Amp. 2. to the optical phase channel. This buffer amplifier has a unity gain and a very high input impedance. The circuit diagram of this buffer amplifier is shown in Fig. C.2.

In the quadrature amplitude channel, we have a low pass filter (LPF1) immediately after the subtractor. LPF1 is a Mini-Circuits BLP-10 low pass filter with a bandwidth of 10 MHz. This is to filter out the locking signal of the system. Depending on the RF locking sideband, the corner frequency of this low pass filter should be adjusted. Isolation transformers (IT's) were used to eliminate 50 Hz signal entering the measurement system. This 50 Hz residual signal is due to the power supply for the ±15V bias voltage of the op-amp's (this is the case for the experiments described in chapter 7). Should the filtering of bias voltages be good enough, we don't need these isolation transformers,

---

rate single-pass measurements for the two mirrors.

Figure 5.4: The electronic configuration of optical homodyne tomography. The arm with Amp.2 is the optical phase channel and the arm with Amp.1 is the quadrature amplitude channel. The mixer is used for demodulation of the difference photocurrent ($i_1 - i_2$). Please refer to the text part for a detailed description of the circuit. LPF: low pass filter; IT: isolation transformer; Amp.: amplifier.

and this is the case for the experiments described in chapter 6 and chapter 8. We used the Mini-Circuits FTB-1-1 as the IT's. The working bandwidth of this IT is 0.2-500 MHz. LPF2 is a simple homemade low pass filter with a corner frequency of 50 kHz (see Fig. 5.5). Before the signal was acquired by the I/O board, we employed an amplifier (Amp. 1) with a gain of 100. The circuit diagram of Amp. 1 is shown in Fig. C.3

Figure 5.5: Low pass filter with a corner frequency of 50 kHz.

The I/O board we used is a 16 bits E Series multifunction I/O board for PCI (PCI-MIO-16XE-50, National Instruments Inc.). We can have multi-channel data acquisition using this board. Technical details of this board are listed in Table 5.3. Please refer to Appendix D for the details of channel connection. This board served very well for clas-

sical state tomography. However, due to its normal sampling rate, we could not use this board in the tomography of a squeezed vacuum state. The phase instability of our squeezed vacuum prevented us from carrying out long term (90 seconds) data acquisition, otherwise the squeezing could have been averaged out by the fluctuating optical phase.

Analog input	16 single-ended, 8 differential channels
	16-bit resolution (1 in 65,536)
	20 kSa/s sampling rate
	DC coupled
Amplifier characteristics	input impedance: 7 GΩ in parallel with 100 pF (normal, powered on)
	input bias current: ± 10 nA
	input offset current: ± 20 nA
Dynamic characteristics (bandwidth)	63 kHz (Gain=1, 2)
	57 kHz (Gain=10)
	33 kHz (Gain=100)

Table 5.3: Specifications of the PCI-MIO-16XE-50 I/O board.

In the case of a squeezed vacuum tomography, we used a high sampling rate HP54820A Infiniium oscilloscope. It has 2 channels, 500 MHz bandwidth, and 2 GSa/s of sampling rate. The memory depth of this oscilloscope is 32,768 points/channel. Details of data processing in this case will be discussed in chapter 8.

A device that was frequently used in the data acquisition of optical phase channel is a low noise pre-amplifier (Model SR560) from the Stanford Research Systems Inc.. It has full range of gain and filter corner frequency for flexible adjustment in different experimental cases.

Another electronic element that's used in the classical state tomography is the voltage-controlled attenuator (VCA). We used a Mini-Circuits ZAS-3 in the switching of phase modulation signal (see chapter 7). As shown in Fig. 5.6, modulation signal is applied

Figure 5.6: Voltage controlled attenuator (VCA).

to "IN", a much lower frequency (e. g. 200 Hz) gating signal is applied to "CON", a switched modulation signal is produced in the "OUT" channel. The suppression ratio of

this VCA is at least 48 dB.

5.6 Summary

In this chapter we summarised some optics and electronics which are necessary in a typical OHT experiment. We presented the technical aspects of the photodetector, the diode pumped Nd:YAG laser and the high finesse optical cavity. Details were given for the characterisation of a high finesse optical cavity, which will be used in our laser phase noise measurement experiment. The electronic configuration of OHT experiment was discussed and some practical problems addressed. We also explained the use of an I/O board, which is a key device in the data acquisition. We will refer to these technical details from chapter 6-8, where we will demonstrate OHT for different quantum and classical states.

Chapter 6

Laser phase noise measurement

6.1   Introduction

As a powerful tool in physics (actually, science), the laser is now widely used in many different applications. There are many different kinds of lasers and they have very different features. In this chapter we concentrate on the noise measurement of a diode pumped Nd:YAG laser, but we believe our method is quite general and can be applied to the study of other laser systems. Although a laser beam can be approximated as monochromatic radiation, with a very well defined frequency, this is not true in the precise sense. In quantum optics, we use RF sidebands to describe the fluctuations of a laser. A realistic laser always has intensity and frequency fluctuations, and these can be described as RF sidebands. We can use an RF spectrum analyser to measure the amplitude sidebands of a laser and get the noise equivalent power of the fluctuations.

   Here we are particularly interested in the phase noise measurement of a laser. The phase noise of a laser is of concern when we want to apply phase or frequency modulation and carry out measurements of these properties. However, it is much harder to determine the phase noise than the intensity noise of a laser. The intensity noise can be measured directly by a photodiode and then analysed by an RF spectrum analyser. This is not true for the phase noise. There have been many reports of laser frequency stabilisation, and the reduction of laser phase noise [89–94]. Similarly there are reports on measuring the linewidths of lasers [95–98]. However, to the best of our knowledge, there is no investigation yet for the full RF phase noise spectrum.

   In this chapter we present an alternative method for characterising the phase noise of a diode pumped Nd:YAG laser. It is based on the optical homodyne tomography (OHT). In contrast to conventional homodyne measurement, which normally gives us the variance of a particular quadrature, OHT can give us the probability density functions (PDF's) for all quadratures. In particular, we can measure the PDF of the phase quadrature, and thus determine the phase noise of the laser.

6.2   Can we measure the phase noise with a Mach-Zehnder interferometer?

In order to do OHT, we need a strong local oscillator which is phase coherent with the laser beam which we want to measure. One way is to phase lock two lasers together and use the output of the laser with high power as the local oscillator [99, 100]. This is very complicated technology. A simpler idea is to use a Mach-Zehnder interferometer as shown in Fig. 6.1. We can use the unbalanced beam splitter M1 to split the power of a laser beam, and use the strong arm as a local oscillator. If this would work, it could be a really simple method. Here we provide a calculation. Unfortunately the result shows that it's impossible to measure the laser noise by this configuration.

Figure 6.1: Mach-Zehnder interferometer and the measurement of laser noise. $V_{1L}(\omega)$, $V_{2L}(\omega)$: amplitude and phase quadrature variances of the laser; $\hat{a}_L$, $\hat{b}$, $\hat{a}_I$, $\hat{a}_{II}$, $\hat{a}_A$, $\hat{a}_B$: annihilation operators of the corresponding fields; $\varepsilon_1 = 0.95$: power reflectance of the beam splitter M1; $\varepsilon_4 = 0.5$: power reflectance of the beam splitter M4; PDA, PDB: photodetectors.

As shown in Fig. 6.1, $\hat{a}_L$ and $\hat{b}$ are the annihilation operators of the laser field and the vacuum port respectively. $\hat{a}_I, \hat{a}_{II}, \hat{a}_A$ and $\hat{a}_B$ are also annihilation operators of the corresponding fields. Suppose we can vary the phase of arm II by a piezo-mounted mirror M3. ε_1 and ε_4 are the power reflectance of the beam splitters M1 and M4 respectively. Suppose $\varepsilon_1 = 0.95$ and $\varepsilon_4 = 0.5$. Most of the laser power is reflected into arm II as a local oscillator, and arm I has a weaker beam. We have two photodetectors PDA and PDB for a balanced homodyne measurement. For M1, we can write down the relationship of the operators [101],

$$\begin{pmatrix} \hat{a}_I \\ \hat{a}_{II} \end{pmatrix} = \begin{pmatrix} \sqrt{1-\varepsilon_1}e^{i\phi_{\tau 1}} & \sqrt{\varepsilon_1}e^{i\phi_{\rho 1}} \\ -\sqrt{\varepsilon_1}e^{-i\phi_{\rho 1}} & \sqrt{1-\varepsilon_1}e^{i\phi_{\tau 1}} \end{pmatrix} \begin{pmatrix} \hat{a}_L \\ \hat{b} \end{pmatrix}. \tag{6.1}$$

For M4, we choose a fixed matrix (note that $\varepsilon_4 = 0.5$),

$$\begin{pmatrix} \hat{a}_A \\ \hat{a}_B \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_{II}e^{i\phi_{II}} \\ \hat{a}_I \end{pmatrix}. \tag{6.2}$$

From Eq. (6.1) and Eq. (6.2) we can express the output operators by the input operators,

$$\begin{aligned} \begin{pmatrix} \hat{a}_A \\ \hat{a}_B \end{pmatrix} &= \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{i\phi_{II}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\sqrt{\varepsilon_1}e^{-i\phi_{\rho 1}} & \sqrt{1-\varepsilon_1}e^{i\phi_{\tau 1}} \\ \sqrt{1-\varepsilon_1}e^{i\phi_{\tau 1}} & \sqrt{\varepsilon_1}e^{i\phi_{\rho 1}} \end{pmatrix} \begin{pmatrix} \hat{a}_L \\ \hat{b} \end{pmatrix} \\ &= \frac{\sqrt{2}}{2} \begin{pmatrix} -\sqrt{\varepsilon_1}e^{i\phi_{II}-i\phi_{\rho 1}} + \sqrt{1-\varepsilon_1}e^{i\phi_{\tau 1}} & \sqrt{1-\varepsilon_1}e^{i\phi_{II}-i\phi_{\tau 1}} + \sqrt{\varepsilon_1}e^{i\phi_{\rho 1}} \\ \sqrt{\varepsilon_1}e^{i\phi_{II}-i\phi_{\rho 1}} + \sqrt{1-\varepsilon_1}e^{i\phi_{\tau 1}} & -\sqrt{1-\varepsilon_1}e^{i\phi_{II}-i\phi_{\tau 1}} + \sqrt{\varepsilon_1}e^{i\phi_{\rho 1}} \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} \hat{a}_L \\ \hat{b} \end{pmatrix}. \tag{6.3}$$

We note that $\phi_{\rho 1}, \phi_{\tau 1}$ and ϕ_{II} can be chosen independently. We can rewrite this relationship as

$$\begin{aligned} \hat{a}_A &= A_{11}\hat{a}_L + A_{12}\hat{b}, \\ \hat{a}_B &= A_{21}\hat{a}_L + A_{22}\hat{b}, \end{aligned} \tag{6.4}$$

where we have defined these four coefficients

$$\begin{aligned} A_{11} &= \frac{\sqrt{2}}{2}(-\sqrt{\varepsilon_1}e^{i\phi_{II}-i\phi_{\rho 1}} + \sqrt{1-\varepsilon_1}e^{i\phi_{\tau 1}}), \\ A_{12} &= \frac{\sqrt{2}}{2}(\sqrt{1-\varepsilon_1}e^{i\phi_{II}-i\phi_{\tau 1}} + \sqrt{\varepsilon_1}e^{i\phi_{\rho 1}}), \\ A_{21} &= \frac{\sqrt{2}}{2}(\sqrt{\varepsilon_1}e^{i\phi_{II}-i\phi_{\rho 1}} + \sqrt{1-\varepsilon_1}e^{i\phi_{\tau 1}}), \\ A_{22} &= \frac{\sqrt{2}}{2}(-\sqrt{1-\varepsilon_1}e^{i\phi_{II}-i\phi_{\tau 1}} + \sqrt{\varepsilon_1}e^{i\phi_{\rho 1}}). \end{aligned} \tag{6.5}$$

Now let's first look at the photon number variance of a single detector, say, PDA. The photon number operator is defined as

$$\hat{N}_A = \hat{a}_A^\dagger \hat{a}_A. \tag{6.6}$$

From Eq. (6.3) we can straightforwardly evaluate the photon number variance

$$\begin{aligned} V_{NA} &= \langle \hat{N}_A^2 \rangle - \langle \hat{N}_A \rangle^2 \\ &= \frac{1}{4}[1 - 2\sqrt{\varepsilon_1(1-\varepsilon_1)}\cos(\phi_{II} - \phi_{\rho 1} - \phi_{\tau 1})]^2 V_{NL} \\ &\quad + \frac{1}{4}[1 - 4\varepsilon_1(1-\varepsilon_1)\cos^2(\phi_{II} - \phi_{\rho 1} - \phi_{\tau 1})]N_L, \end{aligned} \tag{6.7}$$

where $N_L = \langle \hat{a}_L^\dagger \hat{a}_L \rangle$ is the average photon number of the laser field and

$$V_{NL} = \langle (\hat{a}_L^\dagger \hat{a}_L)^2 \rangle - \langle \hat{a}_L^\dagger \hat{a}_L \rangle^2 \tag{6.8}$$

is the variance of the photon number of the laser field. When $\varepsilon_1 \to 1$,

$$V_{NA} = \frac{1}{4}(V_{NL} + N_L). \tag{6.9}$$

From Eq. (6.7) and Eq. (6.9) we know that the direct noise measurement of a single detector can not provide any information about the laser phase noise. Instead, it provides information about the intensity noise, which is not surprising. Now let's look at the difference-current measurement of the two detectors. First let's introduce the linearisation procedure. For a general annihilation operator $\hat{a}$,

$$\hat{N} = \hat{a}^\dagger \hat{a} \tag{6.10}$$

is the photon number operator. We can also define the rotated-quadrature operator

$$\hat{X}(\theta) = (\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta})/2, \tag{6.11}$$

so that

$$\hat{X}_1 = \hat{X}(0) = (\hat{a} + \hat{a}^\dagger)/2 \tag{6.12}$$

and

$$\hat{X}_2 = \hat{X}(\pi/2) = (\hat{a} - \hat{a}^\dagger)/2i \tag{6.13}$$

are the amplitude and phase quadrature operators respectively. If we write

$$\hat{a} = \alpha + \delta\hat{a}, \tag{6.14}$$

where $\alpha = \langle \hat{a} \rangle$ is the average value of $\hat{a}$, we can focus on $\delta\hat{a}$ and omit some higher orders of $\delta\hat{a}$ in our calculation. This is the basic idea of the linearisation approximation [80–83]. Now let's assume

$$\hat{a}_L = \alpha + \delta\hat{a}_L, \tag{6.15}$$

where $\alpha = |\alpha|e^{i\phi}$ is the average value. We can evaluate the photon number variance of the laser,

$$\begin{aligned}
V_{NL} &= Var(\hat{a}_L^\dagger \hat{a}_L) \\
&= Var[(\alpha^* + \delta\hat{a}_L^\dagger)(\alpha + \delta\hat{a}_L)] \\
&\approx Var(\alpha^*\delta\hat{a}_L + \alpha\delta\hat{a}_L^\dagger) \\
&= Var[2|\alpha|\delta\hat{X}_L(\phi)] \\
&= 4|\alpha|^2 Var[\delta\hat{X}_L(\phi)].
\end{aligned} \tag{6.16}$$

For the average photon number,

$$N_L = \langle \hat{a}_L^\dagger \hat{a}_L \rangle = |\alpha|^2. \tag{6.17}$$

For the vacuum state $|0\rangle$,

$$\hat{X}_v(\theta) \equiv \frac{1}{2}(\hat{b}e^{-i\theta} + \hat{b}^\dagger e^{i\theta}), \tag{6.18}$$

$$\delta\hat{X}_v(\theta) = \hat{X}_v(\theta), \tag{6.19}$$

$$\langle \hat{X}_v^2(\theta) \rangle = 1/4, \tag{6.20}$$

$$\langle \hat{X}_v(\theta) \rangle = 0, \tag{6.21}$$

$$Var[\delta\hat{X}_v(\theta)] = Var[\hat{X}_v(\theta)] = 1/4. \tag{6.22}$$

From Eq. (6.22) we know that the quadrature amplitude variance of the vacuum state is independent of the quadrature angle, θ, so we can generally define

$$Var[\delta\hat{X}_v] \equiv Var[\delta\hat{X}_v(\theta)] = 1/4. \tag{6.23}$$

Eq. (6.9) can now be expressed as

$$V_{NA} = |\alpha|^2 Var[\delta\hat{X}_L(\phi)] + |\alpha|^2 Var[\delta\hat{X}_v]. \tag{6.24}$$

This simply indicates the photon number variance detected by PDA consists of two parts,

one originates from the laser itself and the other one is from the vacuum. What if we make a difference-current measurement? From Eq. (6.4) and Eq. (6.5), using the linearisation approximation, we can calculate the difference-current operator

$$
\begin{aligned}
\hat{i}_- &\equiv \hat{i}_A - \hat{i}_B = \hat{a}_A^\dagger \hat{a}_A - \hat{a}_B^\dagger \hat{a}_B \\
&\approx -\varepsilon_1 (e^{-i\phi_{II}+2i\phi_{\rho 1}}\hat{a}_L^\dagger \hat{b} + e^{i\phi_{II}-2i\phi_{\rho 1}}\hat{b}^\dagger \hat{a}_L).
\end{aligned}
\tag{6.25}
$$

If we treat $\hat{a}_L$ classically and replace $\hat{a}_L$ by $\alpha = |\alpha|e^{i\phi}$, define $\phi_3 \equiv \phi_{II} - 2\phi_{\rho 1}$ and $\phi_4 \equiv \phi_3 + \phi = \phi_{II} - 2\phi_{\rho 1} + \phi$, the value of the difference-current operator $\hat{i}_-$ can be further simplified to give

$$
\begin{aligned}
\hat{i}_- &= -\varepsilon_1 |\alpha| [e^{-i\phi_4}\hat{b} + e^{i\phi_4}\hat{b}^\dagger] \\
&= -2\varepsilon_1 |\alpha| \hat{X}_v(\phi_4).
\end{aligned}
\tag{6.26}
$$

From Eq. (6.26) we know that the difference-current is a measurement of the vacuum noise. No information of the laser phase noise can be extracted this way. We have to include a component into the interferometer which separates off the laser noise (including the phase noise), which we wish to study and allows us to observe this in our experiment.

6.3   Optical cavity as a low pass filter

From the RF sideband point of view, an optical cavity is a low pass filter. According to the input-output theory developed by Collett and Gardiner [79], we can easily verify this statement.

Figure 6.2: Input-output relation of a two-sided cavity. T_1, T_2: the power transmittance of mirrors; $\hat{a}_{in}, \hat{b}_{in}, \hat{a}_{out}, \hat{b}_{out}$: the annihilation operators of the input and output fields; $\hat{a}$: the annihilation operator of the intra-cavity field.

Shown in Fig. 6.2 is a two-sided optical cavity. T_1 and T_2 are the power transmittance of the cavity mirrors. $\hat{a}_{in}, \hat{b}_{in}, \hat{a}_{out}, \hat{b}_{out}$ are the annihilation operators of the input and output fields, while $\hat{a}$ is the annihilation operator of the intra-cavity field. We can write down the input-output relation for these operators [79, 83, 102] as

$$
\hat{a}_{out}(\omega) = \frac{[\frac{\gamma_1 - \gamma_2}{2} + i(\omega - \omega_0)]\hat{a}_{in}(\omega) + \sqrt{\gamma_1\gamma_2}\hat{b}_{in}(\omega)}{\frac{\gamma_1 + \gamma_2}{2} - i(\omega - \omega_0)},
\tag{6.27}
$$

where $\gamma_i = T_i c/p$ are the round-trip loss rates of the mirrors, $p = 2l$ is the round-trip length of the cavity and c is the speed of light. $\omega/2\pi$ denotes the optical frequency. In the

case of symmetric mirrors, $T_1 = T_2 \equiv T$, $\gamma_1 = \gamma_2 \equiv \gamma$, so Eq. (6.27) can be simplified as

$$\hat{a}_{out}(\omega) = \frac{i(\omega - \omega_0)\hat{a}_{in}(\omega) + \gamma \hat{b}_{in}(\omega)}{\gamma - i(\omega - \omega_0)}. \qquad (6.28)$$

For the close to resonance cases, $|\omega - \omega_0| \ll \gamma$,

$$\hat{a}_{out}(\omega) \approx \frac{\gamma \hat{b}_{in}(\omega)}{\gamma - i(\omega - \omega_0)}, \qquad (6.29)$$

the cavity is a through-pass Lorentzian filter. However, for large frequencies, $|\omega - \omega_0| \gg \gamma$,

$$\hat{a}_{out}(\omega) = -\hat{a}_{in}(\omega), \qquad (6.30)$$

they are directly reflected off the cavity. Eq. (6.29) and Eq. (6.30) show that the optical cavity transmits the low frequency RF noise and reflects the high frequency RF noise. This property of optical cavities is perfect for our purpose of measuring the noise of a laser beam. We can build an optical cavity which has a very high finesse $\mathcal{F}$ (so as to reduce the FWHM) and make the transmission efficiency high enough[1] (see section 5.4 for discussion). When we pass a laser beam through this cavity, most of the power is transmitted and a weak beam is reflected. According to the discussion above, for the frequencies above the linewidth of the cavity, the transmitted beam is quantum noise limited (vacuum noise reflected), while the reflected beam has the same noise properties as the incident laser beam (laser noise reflected). On one hand, we separated the laser noise and get a strong quantum noise limited local oscillator. On the other hand, these two beams are still coherent with each other. This is a perfect configuration for OHT of the laser noise. We note that in the Mach-Zehnder setup, a strong local oscillator was obtained, but the laser noise is not separated. Instead, it's mixed up with the vacuum noise by the first beam splitter M1 (see section 6.2) and can not be measured. The additional requirement here is that we have to keep the cavity on resonance, otherwise the reflected quadrature will be rotated and we will start to mix intensity and phase noise.

6.4   Experimental setup

The experimental arrangement for measuring the laser noise is shown in Fig. 6.3. We used a three mirror ring cavity as discussed in section 5.4 (No. 3 cavity). A diode pumped Nd:YAG laser (Lightwave 122) was used and its noise properties studied. Because of the ellipticity (1.3:1) of its spatial TEM00 mode, we used a pair of cylindrical lenses for correction. These lenses have focal lengths of 150 mm and 200 mm respectively (CKX150 AR.33 and CKX200 AR.33, Newport Inc.). The reflected beam from the cavity is very weak (9 % of the incident power[2]) and the transmitted beam is close to a perfect circular TEM00 mode. If the incident beam isn't corrected to a circular beam, the mode matching between the reflected and transmitted beams will be very poor and severely affect the homodyne efficiency. The incident power was reduced to 3 mW to avoid photode-

---

[1]We require a transmission efficiency of at least 50 %. The one we obtained in the experiment was 72 %.

[2]According to the formula in section 5.4 and the specification in table 5.2, the reflection efficiency P_r/P_i should be 1.8 %. We think this discrepancy is due to the assumption that $T_1 = T_2$ in the specification.

Figure 6.3: Experimental arrangement for the laser noise measurement. CL: cylindrical lens; MML: mode matching lens; EOM: electro-optic modulator; PBS: polarising beam splitter; NPBS: non-polarising beam splitter; PZT: piezo-electric transducer; PD: photodetector; LPF: low-pass filter.

tector saturation[3]. The electro-optic modulator (Lasermetric 1039DFW-1000) was used to modulate the laser beam. This is done either for locking the laser to the cavity (or vice versa) or simply for calibration purposes. The laser beam was carefully mode matched to the cavity mode to ensure high transmission of power through the cavity. We achieved a mode matching efficiency of $\eta_{mm} = 96\%$[4]. The ring cavity has a different finesse for s- and p- polarisations due to the polarisation dependent reflectance of the cavity mirrors. In the case of p-polarisation, the finesse is low ($\mathcal{F}_p = 200$), but the transmission efficiency is very high ($T_{eff}(p) = 99\%$). We used s-polarisation in the experiment in order to measure lower RF sidebands. Here the finesse $\mathcal{F}_s$ is 4,000 and the cavity FWHM is 179 kHz. This means we can measure the noise sidebands from ~200 kHZ onwards. The transmission efficiency $T_{eff}(s)$ is 72%. This is very close to the calculated value 75 %. (Please refer to the theory in section 5.4.) For the s-polarised incident beam, because of the much higher circulating power inside the cavity, the lost power due to the rear mirror (PZT mounted) is much larger (25 times as that of p-polarisation). This reduced the transmission efficiency. Part of the reflected beam was split off and detected by PD3 for frequency locking, and the rest was the signal beam to be measured. The transmitted

---

[3]This is due to the saturation of our amplification circuit, rather than the photodiode itself.

[4]We scanned the cavity and measured the power spectrum of the reflected light with a high speed oscilloscope (HA54820A Infiniium). η_{mm} was determined by the relative amplitude of the TEM00 mode with respect to the whole spectrum.

beam was mixed with the signal beam by a 50:50 non-polarising beam splitter (NPBS) and its phase scanned by a PZT. We used a matched pair of Epitaxx ETX500 InGaAs photodiodes in combination with transimpedance RF amplifiers (CLC 420, CLC 430) with a 3 dB bandwidth of 20 MHz to detect the light from the NPBS. These customised low noise amplification circuits were built into the photodetectors [84]. The rest of the measurement system is the typical OHT setup as discussed in section 5.5. We scanned the PZT with a function generator at 5 Hz. For every continuous scan of the local oscillator phase, we collected 200,000 data points. This takes about 90 seconds. For these scans we collected data into 120 different bins, for the corresponding phase θ_i and for each value θ_i one PDF is recorded. In this experiment we acquired two channels of data, one channel is the quadrature amplitude, the other channel is the optical local oscillator phase, which is obtained from one of the photodetectors. The interference trace (DC) is a sine of the optical phase θ.

6.5   Results

6.5.1   Amplitude and phase noise spectra

The first kind of measurement we made was conventional homodyne detection. We used an RF spectrum analyser to measure the difference current directly. By manually adjusting the voltage driving the PZT, we were able to minimise or maximise the calibration phase modulation signal, and thus determine the quadrature we were measuring. We show in Fig. 6.4 one of these results. The calibration peak at 12 MHz is a -13 dBm phase modulation signal. When this peak was maximised, we obtained the phase quadrature noise spectrum. On the other hand, when this peak was minimised, we obtained the amplitude quadrature noise spectrum.

Figure 6.4: Amplitude and phase quadrature noise of the laser. A 12 MHz phase signal was used for calibration. RBW: 100 kHz; VBW: 10 kHz.

We note that phase noise dominates for most of the RF regime. Above 10.5 MHz, both the amplitude and phase noise approach the quantum noise limit. In this measurement,

we turned on the built in noise eater of the laser, so we don't see any resonant relaxation oscillation (RRO) in the amplitude noise spectrum. We think the dominating phase noise below 10.5 MHz comes from the technical sources in the laser, e.g., the mechanical and thermal instability of the NPRO.

Shown in Fig. 6.5 is a second set of measurements when we turned off the noise eater of the laser and zoomed into lower frequencies. We see clearly the RRO at 0.53 MHz in the amplitude noise spectrum. This is the only region where the amplitude noise is larger than the phase noise. We recorded the error signal spectrum from the locking loop, which is also a measure of the phase noise[5]. The features of the error signal are similar to that of the phase noise below 0.8 MHz. However, for frequencies higher than 0.8 MHz, the error signal is almost flat. This tells us that the homodyne measurement is more sensitive in detecting the laser noise.

Figure 6.5: Amplitude and phase quadrature noise with the noise eater turned off. Resonant relaxation oscillation (RRO) was at 0.53 MHz in the amplitude noise spectrum. The error signal spectrum was recorded for comparison. RBW: 30 kHz; VBW: 1 kHz.

6.5.2 Pre-stabilisation of the laser

In the previous section, we locked the optical cavity to the laser. In this sense we studied the free-running noise of the laser. The frequency servo we used in that case had a bandwidth of 10 kHz, which was limited by the natural vibrational frequency of the piezo-mirror system. Now we want to do it the other way around. We used another frequency servo to lock the laser to the cavity. Instead of actuating the piezo-mounted mirror of the cavity, we actuated the PZT that was bonded to the laser crystal. This increased the locking bandwidth to 50 kHz [103]. Because the cavity was housed in an aluminum spacer sitting on the optical table, it didn't suffer from severe temperature

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[5]The locking loop has an RF signal at 27.7 MHz for modulation and demodulation. If the laser phase fluctuation is small at one sideband, the demodulated signal after the mixer (the error signal) is also small at that sideband.

drift and hopefully this will improve the noise performance of the laser. In this sense we pre-stabilised the laser frequency.

We show in Fig. 6.6 the result of this locking scheme. One trace is the phase noise spectrum of the free-running laser (notice the peak at 5 MHz is not from the laser itself, but for calibration purposes). The other trace is the phase noise spectrum of the pre-stabilised laser (there is no calibration signal in this case). We see a 10 dB reduction of phase noise below 4 MHz. For higher frequencies, the two traces are much the same. We also retrieved the spectra for amplitude noise, and found the same performance for the two locking arrangements. We conclude that pre-stabilisation by frequency servo only improves the frequency stability[6] of the laser, not the intensity stability.

Figure 6.6: Phase quadrature noise of the pre-stabilised and the free-running laser. A 5 MHz phase signal was used for calibration on the free-running trace. RBW: 100 kHz; VBW: 10 kHz.

6.5.3   Tomographic measurements

In this section we want to demonstrate the tomographic measurement of the laser noise. Instead of setting the local oscillator phase to a particular quadrature as discussed above, we scanned it with the PZT and performed an optical homodyne tomographic measurement of the laser noise. We show in Fig. 6.7 a segment of the raw data that was acquired by the I/O board. The anti-nodes correspond to the phase quadrature and the nodes correspond to the amplitude quadrature. These already suggest that the phase quadrature is much noisier than the amplitude quadrature. Shown in Fig. 6.8 is the reconstructed Wigner function at 0.6 MHz. X_1 is the amplitude quadrature and X_2 is the phase quadrature (this stays the same for all discussions in this thesis). The variance of the phase quadrature is larger than the variance of the amplitude quadrature, and this is consistent with our previous result of conventional homodyne detection (see Fig. 6.5). We note that the reconstructed elliptic Gaussian Wigner function represents a super-Poissonian state

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[6]Frequency is the differentiation of phase with respect to time. Frequency and phase stability are directly related although different.

of the light and is different from a squeezed state Wigner function, which is also elliptic (see chapter 8).

Figure 6.7: A segment of the raw data that was acquired by the I/O board. The anti-nodes correspond to the phase quadrature and the nodes correspond to the amplitude quadrature. a. u.: arbitrary unit.

Another result is shown in Fig. 6.9 for the reconstruction at 4.0 MHz. We see clearly a phase quadrature diffused blob. This is due to the phase signal close to, but not exactly at, 4.0 MHz (within the detection bandwidth of 50 KHz). This can be explained by the theory developed in section 3.6.1 (see Eq. (3.37)). Essentially the phase signal is asynchronously demodulated by the 4.0 MHz demodulation signal and caused an oscillating coherent amplitude in phase space. We note that the phase signal close to 4.0 MHz is generated within the laser system, we did not put it on from the outside. In chapter 7 we will show the effect of deliberately modulating the laser beam and see how the modulation signal can be retrieved by different demodulation techniques.

OHT system has a frequency dependent response. In order to demonstrate this we applied phase modulation to the laser beam at different sideband frequencies. Shown in Fig. 6.10 are contours of Wigner functions reconstructed at different frequencies. One thing we can verify from this result is that the direction of the phase signal (displacement from the origin) is consistent with the phase quadrature noise which is elongated along the same direction. Above 10.5 MHz, both the amplitude and phase noise approach the quantum noise limit (see Fig. 6.4), and the contours of the Wigner functions should have about the same size, but this is not the result we actually got (see the contours for 11-14 MHz in Fig. 6.10). This is clearly a demonstration of the frequency dependence of OHT system for noise sidebands. This also suggests that the quantum noise limit can not be calibrated by the current electronic demodulation system of OHT. Some improvement of the electronic system has to be done in order to match the quality of the sophisticated electronic demodulation system of a commercial spectrum analyser. For example, the fre-

quency response of LPF2 and Amp.1 (see Fig. 5.4) can be calibrated. Another thing is that a different displacement from the origin occurred for different frequencies, even though we applied the same amount of modulation (see contours at 7.7, 8, 13 and 14 MHz). This is due to the frequency dependence of the response of the phase modulator as well as OHT system. However, we note that the electronic demodulation phase in our experiment (ψ in Fig. 3.4) has not been adjusted to optimise the demodulated phase signal, and this should account for most of the frequency dependence of the response. This also caused the changed polarities of the retrieved modulations in phase space.

6.5.4   Interference problem of separated beams

In a normal homodyning we can get very high interference visibility (say, 98%). The interference trace is a smooth sine of the optical phase. However, in the laser noise measurement experiment discussed in this chapter, homodyne interference of the reflected and the transmitted beams of the ring cavity has a different feature. We found that the interference trace is not as smooth as it is in a normal homodyning. There is certain amount of fluctuation on the interference trace. For identical optical power of the two beams (adjusted by inserting a neutral density filter), we did get very high visibility (97.5%) calculated by the minimum and maximum of the interference trace. But, in order to get a reasonably smooth interference trace, which can be acquired as the optical phase, we had to low pass filter the original interference trace. We actually used a low noise pre-amplifier (Model SR560, Stanford Research Systems Inc.) with adjustable filtering corner frequency to smooth the trace. After the filtering, the visibility dropped up to 60% (depending on the filtering corner frequency). We think this decrease in visibility is due to the way we separated the beams. In a Mach-Zehnder interferometer, the two beams are separated by a beam splitter and have the same noise properties, thus they can have very high interference visibility. However, in our current experiment, the reflected beam is much noisier than the transmitted beam, because the former carries the laser noise. These two beams have very different (if not opposite) noise properties. We actually monitored the power of the reflected beam and found that it is far from a flat DC trace[7].

   We show in Fig. 6.11 and Fig. 6.12 the amplitude noise spectra of the reflected and transmitted beams. They were directly measured with one photodetector while the cavity was locked. (Please notice that the amplitude noise in Fig. 6.4 and Fig. 6.5 were measured by homodyning.) Since the reflected and transmitted beams have different power, we normalise the reflected trace with respect to the transmitted trace. The amplitude noise of the laser is quantum noise limited at 10 MHz (see section 6.5.1), and this should also be true for both the reflected and transmitted beams. We therefore shift the reflected traces by 6.9 dB and get the normalised reflected traces. It can be seen that the reflected beam is much noisier than the transmitted beam below 3 MHz.

   We note that we have carefully mode matched the laser to the cavity and this ruled out the possibility of poor spatial mode overlapping. The interference problem can only originate from the temporal properties of the two beams.

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[7]One may argue that since the incident and transmitted power are constant, why should the reflected power be fluctuating? Isn't this violating the conservation of energy? We note that in that sense the reflected power is also constant (we can use the same scale on an oscilloscope to "make" the reflected power "flat"). The real situation is that the reflected power is much lower (9 % of the incident power). The power fluctuation is in this smaller scale. If we look at the incident power in this smaller scale, we can see the same amount of fluctuation as well.

6.6 Conclusions

We used a high finesse cavity as a frequency dependent beam splitter for the light of a diode-pumped Nd:YAG laser. The cavity behaves like an optical low pass filter. The transmitted beam is quantum noise limited for detection frequencies above the cavity linewidth. The reflected beam carries the laser noise and modulation at these frequencies. Conventional homodyne measurement of these two beams gave us the amplitude and phase noise spectra of the laser. The quadrature was adjusted by the driving voltage of the PZT. The error signal spectrum was also recorded to verify the phase noise measurement. We found that the homodyne method is more sensitive than the error signal method. By locking the laser to the cavity, we pre-stabilised the laser frequency and the phase noise was reduced by 10 dB below 4 MHz. We also employed OHT to reconstruct the Wigner functions at different frequencies, and found larger phase quadrature variance than the amplitude quadrature variance, which is consistent with the results of the conventional homodyne detection. We conclude that OHT is a sensitive method in determining the embedded signals in the laser beam. Signals close to demodulation frequency caused the phase quadrature diffusion in phase space. We encountered interference problem which doesn't occur in a normal Mach-Zehnder interferometer. This is due to the different temporal properties of the reflected and the transmitted beams. An optical cavity separates the incident beam very differently compared to a simple beam splitter.

(a)

(b)

Figure 6.8: The reconstructed Wigner function (a) and its contour plot (b) at 0.6 MHz. The phase quadrature (X_2) is noisier than the amplitude quadrature (X_1). incident power: 2 mW. The Wigner function and its contour are normalised by quantum noise limit. This remains to be true for all the experimentally reconstructed Wigner functions in this thesis.

(a)

(b)

Figure 6.9: The reconstructed Wigner function (a) and its contour plot (b) at 4.0 MHz. The phase quadrature (X_2) is diffused due to a phase signal within the detection bandwidth but not exactly at 4.0 MHz. incident power: 2 mW.

Figure 6.10: The frequency dependent response of the optical homodyne tomography system to signal and noise sidebands. The crosses denote the origins of phase space. incident power: 2 mW.

Figure 6.11: Amplitude noise spectra of the reflected and transmitted beams of a ring cavity directly measured with one photodetector. The noise eater of the laser was switched off and there is relaxation oscillation. We shift the reflected trace by 6.9 dB to give the normalised reflected trace with respect to the transmitted trace. It can be seen that the reflected beam is noisier than the transmitted beam below 3 MHz. incident power: 2 mW; RBW: 30 kHz; VBW:1 kHz.

Figure 6.12: Amplitude noise spectra of the reflected and transmitted beams of a ring cavity directly measured with a photodetector. The noise eater of the laser was switched on and there is no relaxation oscillation. We shift the reflected trace by 6.9 dB to give the normalised reflected trace with respect to the transmitted trace. It can be seen that the reflected beam is noisier than the transmitted beam below 3 MHz. incident power: 2 mW; RBW: 30 kHz; VBW:1 kHz.

Chapter 7

Measuring the classical non-Gaussian states

7.1 Introduction

In the previous chapter we discussed measurements of the noise properties of the laser itself. Now we will use optical homodyne tomography (OHT) to measure information carried in the laser beam. In practical applications, such as sensing or communication, a laser beam is modulated in order to carry information. This can occur at different modulation frequencies or channels Ω_m. Each modulation frequency has its own Wigner function $W_{\Omega m}(x_1, x_2)$, and can vary dramatically from channel to channel [15, 79]. Hence, a realistic laser beam with signal modulations is a propagating multi-mode quantum state, which is quite different to the single mode intra-cavity description of an optical state. Such a realistic light beam can only be described by a spectrum of Wigner functions W_Ω, one for each frequency Ω.

In the early experiments [7–9] of pulsed laser OHT, only one single Wigner function was reconstructed. This Wigner function contains all statistical moments of the photon number in the pulses and individual Fourier components of a pulse could not be separated. In more recent experiments with cw lasers, Wigner functions $W_{\Omega d}$ for squeezed and classical states were reported for specific detection frequencies Ω_d. In the case of squeezed light, Wigner functions with elliptical two-dimensional Gaussian quasi-probability distributions were demonstrated [11–14].

In this chapter, we use the Wigner function representation to describe coherent states with different forms of classical sinusoidal phase modulation (PM). At the modulation frequency we observe that the Wigner functions are displaced away from the origin of the phase space and the results allow us to directly determine the signal to noise ratio (SNR) for both amplitude and phase signals[1]. At all other frequencies, the Wigner functions are centred at the origin. We demonstrate how the Wigner functions are controlled by the quadrature and depth of modulation.

One of the advantages of OHT over conventional homodyne measurements [70–72] is that OHT can show any non-Gaussian features in the Wigner functions of more complex optical states. Such features are contained in the higher order moments of the noise statistics with the result that a simple measurement of the noise variance (2nd order moment) alone is an insufficient description. The additional complexity means Wigner function reconstruction demands more sophisticated statistical analysis. Unfortunately, unlike experiments in atom optics [21, 22], complex optical states are extremely difficult to generate [104]. They require highly nonlinear processes which have fluctuations comparable to the average steady state amplitude. To date most of the work in this area are theoretical

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[1]In this case we had the optimal demodulation phase (ψ in Fig. 3.4), in contrast to the case discussed in section 6.5.3.

proposals for the generation of such non-classical states[2] (see section 9.1 for a summary). We test the ability of OHT to reconstruct highly non-Gaussian Wigner functions by using laser beams with time varying modulation to produce classical superpositions of optical states. We discuss the accuracy of the OHT technique and some of the practical limitations.

This work demonstrates the capability of OHT for the diagnosis of realistic laser beams and establishes techniques that will be important once highly non-Gaussian, e. g. "Schrödinger's cat", optical states can be experimentally realised.

7.2 Phase and amplitude modulation

The electric field of a phase modulated laser beam can be written as

$$E_{pm} = E_0 \cos[\omega t + \delta \sin(\Omega_m t)], \tag{7.1}$$

where δ is the modulation index (maximum phase change), Ω_m is the modulation frequency and ω is the carrier frequency. Eq. (7.1) can be expanded into harmonics of the modulation frequency Ω_m [1],

$$\begin{aligned} E_{pm} &= E_0 \left[J_0(\delta) \cos \omega t + J_1(\delta) \cos(\omega + \Omega_m)t - J_1(\delta) \cos(\omega - \Omega_m)t \right. \\ &\quad \left. + J_2(\delta) \cos(\omega + 2\Omega_m)t + J_2(\delta) \cos(\omega - 2\Omega_m)t + \ldots \right], \end{aligned} \tag{7.2}$$

where $J_m(\delta)$ are Bessel functions. For small modulation index $\delta \ll 1$, we can approximate Eq. (7.2) to the first order sidebands and simplify E_{pm} as

$$E_{pm} = E_0[\cos \omega t + \beta \cos(\omega + \Omega_m)t - \beta \cos(\omega - \Omega_m)t], \tag{7.3}$$

where $\beta = J_1(\delta)$ is the modulation depth and $\beta \ll 1$.

Amplitude modulated optical field has the form

$$E_{am} = E_0(1 + \delta \sin \Omega_m t) \cos \omega t, \tag{7.4}$$

where $\delta \ll 1$ is small and causes the amplitude of the electric field to vary. Please refer to Appendix E for a detailed discussion of modulation.

In practice, we use electro-optic modulators to modulate the laser beam. The Pockel's effect of electro-optic crystals such as lithium niobate are used. The refractive index of the crystal can be changed by an externally applied voltage, which in turn causes a phase change of the laser beam passed through the crystal. There are technical limitations to electro-optic modulators (EOM's). Normally there is residual amplitude modulation associated with a phase modulator[3]. On the other hand, residual phase modulation is much smaller in an amplitude modulator. In our experiments, we use modulators from Lasermetric Inc.. The polarisation of the incident beam should be adjusted so that it's s-polarised (i. e. vertically polarised) to minimise the beam walkoff due to the birefringence of the crystal.

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[2]Conditional Fock states are routinely generated in entangled-photon experiments [25, 105–110]. Recently quantum tomography was applied to such single photon Fock state [111].

[3]To achieve purer phase modulation, one should carefully align the beam to the crystal's propagation axis and align the laser's polarisation to the crystal's electro-optically active axis.

7.3   Experimental setup and results for the classical states tomography

7.3.1   The experimental arrangement

The aim of our experiment was two fold: we wished to demonstrate OHT for the case of modulated light fields and we wished to explore the applications and limitations of OHT for signal to noise ratio measurements. As a light source we used a diode pumped Nd:YAG laser (Lightwave 120) with a wavelength of 1064 nm. The power was reduced from 40 mW to 3 mW to avoid photodetector saturation. The setup (see Fig. 7.1) was a Mach-Zehnder interferometer, the commonly used balanced homodyne detector [70–72]. We used two 50:50 non-polarising beam splitters M1 and M4 in the interferometer. By carefully matching the two arms (identical EOM's were inserted, one in each arm to improve the mode matching), we achieved an interference visibility of 99%. Then we inserted a neutral density filter of OD 2 (99% absorption) in one arm and drove the phase electro-optic modulator by an RF signal at $\Omega_m = 7.0$ MHz to produce a low optical power test beam. In the other arm, we used a piezo-mounted reflecting mirror M3, in order to scan the local oscillator phase θ. The power of the RF signals determines the modulation depth $\beta's$. We achieved values of $0 < \beta < 7.1 \times 10^{-3}$. The modulation signals were conveniently changed by an inserted adjustable RF attenuator. The measurement system was the same as discussed in section 6.4. We used a matched pair of Epitaxx ETX500 InGaAs photodiodes to detect the light from the second beam splitter. We mixed down the difference photocurrent of the two photodetectors at the modulation frequency. This was then low-pass filtered and recorded by an I/O board connected with a computer sampling at 20 kHz. These time traces of $i_{\Omega d}(\theta, t)$ were mentioned in section 3.6.1. We scanned the PZT with a function generator at 5 Hz. For every continuous scan of the local oscillator phase, we collected 200,000 data points. For this data we recorded 120 probability density functions (PDF's) for equally spaced quadratures. These PDF's were used as the input into Eq. (3.15) for Wigner function reconstruction.

7.3.2   Wigner function reconstruction for modulated coherent state

As discussed in section 3.6, phase modulation can be retrieved via the tomographic reconstruction of Wigner functions. The resulting Wigner function $W_{\Omega d}(x_1, x_2)$ of a phase modulated coherent state is shown in Fig. 7.2. Here we want to discuss the meaning of this phase space diagram. For a coherent state the Wigner function is symmetric - with concentric contour lines. For a squeezed state the Wigner function is asymmetric, with elliptical contour lines. For all coherent or squeezed states, without modulation ($\beta = 0$), the Wigner function is centered at the origin. This can be seen from Eq. (3.38) where only the second term contributes in this case. The orientation of the ellipse gives the squeezing quadrature. Since the Wigner function is normalized to the standard quantum limit, its position and size is independent of the optical power. Thus, a squeezed vacuum state has the same Wigner function as a bright squeezed state, provided that $P_{LO} \gg P_{test}$. Here we want to mention the widely used picture of a "ball on a stick" [83, 102] which tries to describe several properties of an optical state simultaneously. The "stick" indicates the average, DC optical power while the "ball" represents high frequency, AC fluctuations. The two types of data have to be measured with different instruments, a power meter and an RF photodetector. Experimentally they are completely different, but for conve-

Figure 7.1: Experimental arrangement for optical homodyne tomography of classical states. EOM: electro-optic modulator; ND: neutral-density filter; PZT: piezo-electric transducer; PD: photodetector; LPF: low-pass filter; VCA: voltage controlled attenuator; SA: spectrum analyser. For conventional OHT with synchronous detection only one frequency Ω_m is used. In order to generate classical non-Gaussian states two components are added: the modulation is switched and a different demodulation frequency is used (dashed boxes). [Signal generators with frequencies Ω_m and Ω_d are not used together.]

nience they are shown together in theoretical diagrams. However, the Wigner function $W_{\Omega d}$ which can be measured at a particular frequency Ω_d depends only on the noise and signals at that frequency. Hence, Wigner functions for realistic light beams can only be displaced from the origin by introducing a modulation.

The distance of the centre of the Wigner function from the origin is a direct measurement of the modulation depth. Amplitude modulation causes displacement along the X_1 axis whilst phase modulation along the X_2 axis. The Wigner function gives us simultaneous information about the signal quadrature, strength and noise. It provides the conventional signal to noise ratio (SNR), where both signal and noise are measured in the same quadrature, as well as the relative size of the noise in the orthogonal quadrature. The later is of interest in applications where some degree of crosstalk between the quadratures is unavoidable. For example, in resonantly locked cavities a small imperfection of the locking can introduce a cavity detuning and thus a mixing of the quadratures.

Figure 7.2: The reconstructed Wigner function of a modulated coherent state. X_1 and X_2 are amplitude and phase quadratures respectively. $X_1 = X_2 = 0.5$ represent vacuum noise. The transition from ripples to flat at $X_2 \approx 2$ is an artifact. (The flat area is added for better visual effect.)

7.3.3 Varying the depth of phase modulation

As discussed above, for a coherent beam without modulation ($\beta = 0$), the Wigner function $W_{\Omega d}$ is centered at the origin of the X_1, X_2 space. Now we should vary the depth of modulation and see what will happen. We used phase modulation and synchronous detection for four different modulation depths. The result is shown in Fig. 7.3. For $\beta = 0$ (no modulation, i.e., a vacuum state) the Wigner function is circular and centred at the origin. As the frequency modulation depth is increased, the Wigner function is displaced along the "X_2" (phase variance) axis. For large modulation, it is apparent that the phase modulation process has introduced some additional amplitude modulation, resulting in the Wigner function being displaced vertically from the X_1 axis (amplitude variance). This is due to the imperfections of the phase modulator, a common problem with most EOM modulators.

It is important to note that synchronous demodulation requires the optimisation of the demodulation phase ψ. This, by maximising the detected signal amplitude, ensures that the modulation component is detected with maximum efficiency and results in the optimum SNR being recorded on the Wigner function. We also noticed the elongated contours for larger modulations, this is due to the increased phase uncertainty of modulated states produced by larger modulation signals.

Figure 7.3: Phase modulation with variable depth of modulation. The four Wigner function contours shown in one figure. The contours change from symmetric to non-symmetric when the modulation is increased. X_1 and X_2 are amplitude and phase quadratures respectively. incident laser power (on M1, see Fig. 7.1): 3 mW (we note that the result is independent of the incident laser power); modulation frequency: 7 MHz; RF power driving the modulator: 0 dBm, 5 dBm, 10 dBm and 15 dBm respectively (from right to left).

7.3.4 Switched phase modulation

In order to demonstrate the ability of our OHT system to record the details of a highly non-Gaussian distribution, we added a low frequency modulation to gate the phase modulation on and off. This was done by a voltage controlled attenuator (discussed in section 5.5). The selected gating frequency was 200 Hz, which was within the detection bandwidth (50 kHz) of our detectors (low pass filter followed by I/O board). For this modulation arrangement, we had PM for half of the measurement time and had no PM for the other half of the measurement time. The resulting PDF is shown in Fig. 7.4. The straight trace corresponds to the vacuum state and the curved trace corresponds to the coherent state which was produced by phase modulation. These two traces form two independent Gaussian components for each quadrature. In the amplitude quadrature, they are totally overlapped. In the phase quadrature, they are maximally separated. The corresponding Wigner function is plotted in Fig. 7.5. As can be seen, the gating process (square wave signal generator in Fig. 7.1) produced a Wigner function with two peaks: one peak corresponds to zero modulation and is located at the origin, whilst the other corresponds to phase modulation.

We note that such produced non-Gaussian two-peak Wigner function represents a purely classical mixture state. There is no coherence between the two Gaussian compo-

Figure 7.4: The probability density function resulting from switched phase modulation. unit of the L. O. angle: radian.

nents. In contrast, the subtlety of a "Schrödinger's cat" state is that the two Gaussian components therein are coherently superposed and interference occurs in between [61].

7.3.5   Asynchronous detection

In practical applications, we may not be able to locate the modulation frequency exactly. (We need to match both the frequency and the phase in order to do so.) In this section we demonstrate the results of asynchronous detection. This was achieved experimentally by using a separate frequency Ω_d as the demodulation signal, which was different from the modulation frequency Ω_m used to drive the EOM. It is necessary to ensure that Ω_m and Ω_d differ in frequency by an amount small compared with the detection bandwidth (50 kHz). Otherwise they belong to different channels and the modulation signal cannot be detected. We note that this asynchronous scheme is equivalent to randomised demodulation, the detected Wigner function represents the weighted average of all possible demodulation phase values ψ. The recorded PDF is shown in Fig. 7.6. The distribution for phase quadrature is extremely wide. This is due to the phase quadrature diffusion caused by the random demodulation phase ψ. The resulting Wigner function, for phase modulation, is shown in Fig. 7.7. The distribution is centred at the origin and is spread symmetrically along the phase quadrature axis. The peaks at the extreme of the modulation correspond to the turning points where the dwell time of the modulation, as a function of phase, is greatest. This result is consistent with Eq. (3.37) where $\delta(\Omega_d, \Omega_m)$ representing an oscillating term for asynchronous demodulation. We call this a phase quadrature diffused state. It's analogous to a classical harmonic oscillator, although the

oscillation is now in phase space. This result shows that OHT is extremely sensitive in discriminating signals.

In section 7.3.4, we switched the intensity of the phase modulation on/off and created a two-peak Wigner function which is an overlap of two different Gaussian components. In the asynchronous detection scheme, by continuously changing the demodulation phase ψ, we effectively overlapped many Gaussian components corresponding to different coherent amplitudes. This can be regarded as an extension of the two-peak case. However, the techniques used in these two experiments are very different.

7.4   Conclusions

In this chapter, we have demonstrated the reconstruction of Wigner functions of information carrying laser beams. Phase modulation was used to shift the Wigner function away from the origin of phase space. OHT turned out to be a very good method for the simultaneous determination of signal and noise of a given state. The displacement of the Wigner function $W_{\Omega d}$ provides information on the quadrature and strength of the modulation and the width of $W_{\Omega d}$ describes the noise. The two dimensional signal to noise ratio at any given quadrature can be read directly. At a given RF sideband, we varied the modulation depth and obtained a series of Wigner functions with different displacement from the origin. For larger phase modulation, we observed residual amplitude modulation which was caused by the imperfections of the phase modulator (a common problem with electro-optic phase modulators). This confirmed the sensitivity of OHT in distinguishing signals from noise.

In general, most optical states which can be generated can be described by a linearised theory and thus has a two dimensional Gaussian Wigner function $W_{\Omega d}(x_1, x_2)$. In order to demonstrate the ability of measuring highly non-Gaussian Wigner functions, we first switched the modulation on and off, giving a classical mixture of a vacuum state and a coherent state. The resulted Wigner function has two peaks. In another scheme, we used asynchronous demodulation scheme to detect the optical state, the phase quadrature was diffused and another highly non-Gaussian Wigner function was reconstructed. This diffusion in phase space is analogous to a classical harmonic oscillation.

(a)

(b)

Figure 7.5: The reconstructed Wigner function and its contour plot of a classical mixture state. (a) Wigner function; (b) contour plot.

Figure 7.6: The probability density function resulting from an asynchronous detection. unit of the L. O. angle: radian.

(a)

(b)

Figure 7.7: The reconstructed Wigner function and its contour plot by using a phase-unlocked scheme. (a) Wigner function; (b) contour plot.

Chapter 8

Measuring the squeezed vacuum state

8.1   Introduction

After discussing classical state measurement via optical homodyne tomography (OHT) in chapter 7, we will discuss its application in quantum state measurement in this chapter. Quantum state OHT is more difficult than classical state OHT, because quantum features are normally fragile and sensitive to noise. Quantum features can easily be embedded in noise from the measurement system or smeared out by low detection efficiency.

To date most OHT experiments have been devoted to quadrature squeezed state measurement. The quantum feature for this kind of state is the reduction of the noise level below quantum noise limit in some quadratures. However, this doesn't exclude the possible applications of OHT for other quantum optical states, such as intensity correlated twin beams [112–114] generated from nondegenerate optical parametric oscillators (OPO) above threshold, or strongly correlated photon pairs from sub-threshold OPO. The latter is often used [25, 105–110] to test Bell's theorem [115, 116] or even used in quantum teleportation [117, 118].

In our quantum state OHT, we used the quadrature squeezed vacuum state. Ever since the pioneering experiments in the eighties [56, 119–121], quadrature squeezed states have been successfully generated in many experiments. In the sub-catalog of $\chi^{(2)}$ nonlinear process, squeezed states have been generated from optical parametric oscillators/amplifiers (OPO/OPA) [13, 56, 121–127] and second harmonic generators (SHG) [128–132]. The $\chi^{(2)}$ nonlinear process is very reliable for the two-mode quadrature squeezed state generation. To date all OHT experiments for the quadrature squeezed vacuum states have been limited to the fundamental mode of OPO/OPA, because of difficulty in obtaining optical local oscillators for the second harmonic mode.

In the rest of this chapter, we will discuss our squeezed vacuum OHT experiment and present our result, with calibration by the corresponding vacuum state. Some experimental limits will be addressed. In order to compare with traditional variance measurements performed with spectrum analysers (SA's), we will directly analyse the probability density functions (PDF's) of the states and calculate the higher order moments. We will also analyse the effect of electronic noise on Wigner function reconstruction.

8.2   Experimental setup and results

8.2.1   Setup

The experimental setup for measuring the squeezed vacuum state is shown in Fig. 8.1. The squeezed vacuum was generated from a degenerate type I phase matched sub-threshold OPO (please refer to the paper by Lam et al. [127]). The OPO consisted of

a monolithic lithium-niobate crystal. The OPO was pumped by a second harmonic wave (532 nm) produced by a hemilithic second harmonic generator (SHG). The pump for the SHG was a cw diode pumped Nd:YAG laser (1064 nm, Mephisto700, InnoLight Inc.). Part of the Nd:YAG laser power was split off and passed through a mode cleaner (cavity No. 1 discussed in section 5.4). This beam was used as local oscillator and the power was 2 mW (after the mode cleaner). The OPO was operated below threshold with a green pump power of 300 mW to produce a squeezed vacuum state. For optimum squeezing, the green pump power was 145 mW. We carried out all our measurements at an RF side-band of 3 MHz, which had an optimum degree of squeezing of 7 dB as measured from a spectrum analyser (RBW = 50 kHz, VBW = 1 kHz). The detection system for tomography was the same as that in the classical state experiment except that we used a high speed digital oscilloscope to collect the photocurrent data instead of the I/O board since the phase of the generated squeezed vacuum state was not stable enough for long term data acquisition (about 90 seconds). For every run of the data acquisition, we obtained 32,768 data points. This long trace is divided into 120 quadratures. In such a trace there is not enough data for the reconstruction of a Wigner function, because the PDF's result-ing from a single trace are not accurate enough to represent the actual quadrature value distributions. We have to combine 11 such traces with identical quadrature angle range $(0 \leq \theta < \pi)$ to get the PDF's of the squeezed vacuum state.

8.2.2   Phase instability and data acquisition

As mentioned above, we could not use our I/O board to acquire the data in this experi-ment, because a sampling time of 90 seconds can totally wash out the squeezing feature. With a much higher sampling rate, this would not be a problem. This problem doesn't exist for a spectrum analyser measurement, because it just requires a single scan of data acquisition. For example, the measured squeezing was 7 dB in such measurement; how-ever, this 7 dB of squeezing can not last for 90 seconds consistently. The long optical path length of the setup, the temperature stability of the OPO cavity and the stability of the laser all affect the phase stability of the generated squeezed vacuum. The 90 seconds of averaging of this 7 dB squeezing can easily be 0 dB and we have no way to extract any of the squeezing feature. As a compromise, we used our HP54820A Infiniium oscilloscope to acquire data. This oscilloscope has two channels, and the sampling rate was 2 GSa/s. The resolution was 12 bits with averaging. The only drawback of this oscilloscope for OHT is its memory depth (32,768 points/channel). We had to carry out multiple scans to produce our PDF's. On one hand, we can choose our best scans for producing the PDF's by looking at the phase quadrature noise. On the other hand, this manual handling of data is not as accurate as a single scan by high speed I/O board. What's more, the nu-merical combination of many scans is very tricky. It can easily degrade the squeezing feature if the phase is not exactly correct. Considering how fragile the quantum features are, we suggest the use of high sampling rate I/O board for this kind of quantum state reconstruction, so that enough data can be acquired in short enough time (comparable with the lifetime of the quantum feature, e.g., squeezing).

8.2.3   Efficiency, filtering and squeezing

For quantum state reconstruction, high detection efficiency is necessary. The homodyne visibility of our measurement was 98% and the photodetector had a quantum efficiency

Figure 8.1: Experimental setup for measuring the squeezed vacuum state. SHG: second harmonic generator; OPO: optical parametric oscillator; MC: mode cleaner; PBS: polarising beam splitter; NPBS: non-polarising beam splitter; DM: dichroic mirror; PZT: piezo-electric transducer; PD: photodetector; LPF: low-pass filter.

of 94%. This finite detection efficiency can be modeled [133] by a beam splitter as shown in Fig. 8.2. The power transmittance of the beam splitter is η. The detected output wave is the signal wave convoluted by the vacuum port. This is a Gaussian convolution. Besides these, we found that the electronic configuration of our OHT system had certain amount of electronic noise. It was 7.2 dB below the shot noise limit. By blocking both the signal wave and the optical local oscillator (see Fig. 8.1), we acquired the electronic noise data. By analysing this data, we found that the electronic noise had a Gaussian distribution. As we will show in section 8.3, this Gaussian electronic noise also has a convolution effect on the PDF's. Besides the finite detection efficiency and the Gaussian electronic noise, which gave smoothed (Gaussian-convoluted) PDF's, there was another filtering effect in the reconstruction algorithm of Wigner function (please refer to section 4.5). This combined convolution effect degraded the observed degree of squeezing. The reconstructed Wigner function is a smoothed version of the original one.

There are very good proposals to compensate for the imperfect efficiency of measurement system and for quantum state estimation [134–138]. Physical constraints such as

Figure 8.2: A beam splitter model of finite detection efficiency. $\hat{a}_s$, $\hat{a}_v$, $\hat{a}_{out}$: annihilation operators of the signal port, the vacuum port and the output port respectively; BS: beam splitter; $\eta(< 1)$: power transmittance of the beam splitter, to account for the finite detection efficiency.

non-negativity of probability and unity of sum of probability were used in order to enhance the accuracy of state reconstruction. However, as indicated by Breitenbach [139], no additional quantum features besides the ones detected by standard reconstruction methods were revealed in his experiments. Once convoluted by noise, the quantum feature is lost forever. In practice, deconvolution is rather difficult to implement due to the noisy nature of experimental data and the error-propagation in the reconstruction procedure. On the experimental side, the quantum efficiency of the photodetector is at the limit of available technology. However, in our experiment it was possible to reduce the electronic noise by carefully arranging the electronic elements. To further reduce the electronic noise, we recommend to increase the optical local oscillator power as well as to increase the saturation power of the photodetectors, so that the electronic noise has less severe effect[1].

8.2.4 Vacuum versus squeezed vacuum

The reconstructed Wigner function of a squeezed vacuum state is shown in Fig. 8.3(a). Clearly the phase quadrature x_2 is squeezed and the amplitude quadrature x_1 is antisqueezed. As we know in the conventional variance measurement by spectrum analyser that measuring the asymmetry alone is not sufficient. Squeezing should always be measured together with the quantum noise limit to calibrate the recorded noise traces. In order to get quantitative information about the squeezed state, we calibrated it with the corresponding vacuum state. This was done by blocking the squeezed vacuum port and repeating the measurement. The Wigner function of the vacuum state is shown in Fig. 8.3(b). While acquiring data for the squeezed vacuum, we couldn't extract an optical phase reference from the interference trace as we do in the classical OHT, because there is no such trace while the local oscillator is mixing with a vacuum (squeezed). Fortunately we have a compromise. We can locate the maximum and minimum of the quadrature

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[1]To achieve these we can choose photodiodes that can handle more optical power and modify the amplification circuits.

amplitude variance for each scan and determine the section of data we should use for producing PDF's. This is a working, although time-consuming, method.

We show the contour plot of the Wigner function of the squeezed vacuum in Fig. 8.4. For an ideal Gaussian Wigner function, we could measure the degree of squeezing from any one of the contour lines and get the same result. However, we found that the degree of squeezing and anti-squeezing measured at different heights of the Wigner function are different. These results are shown in Fig. 8.5(a). Here the size of the contour, normalised to the same contour of the vacuum state, is evaluated in [dB] as a function of the height (0 to 1) of the contour[2]. We got larger squeezing and anti-squeezing from lower heights on the Wigner function. Conventionally for one type of Wigner function squeezing is measured at $1/e^2$ of the peak height and we obtained a squeezing of -4.1 dB and an anti-squeezing of +8.7 dB. The degradation of squeezing (from -7 dB to -4.1 dB) was mainly due to the electronic noise. Please refer to section 8.3 for a discussion. We also show in Fig. 8.5(b) the calibrated uncertainty area as a function of the height (0 to 1) of the contour. Only at around 0.7 of the peak height does the uncertainty area reach a minimum of 1.2. This plot does not support the assumption that the reconstructed state is a minimum uncertainty state, in contrast to the data obtained with conventional variance measurements. From all these results we know that the reconstructed Wigner function is not a perfect Gaussian function. However, considering the lower accuracy of Wigner functions reconstruction in comparison with the conventional variance measurement, we think the generated squeezed vacuum state is still a minimum uncertainty Gaussian state. The problem is with our experimental data acquisition/combination and numerical reconstruction, and we will discuss the data analysis in greater detail below.

8.2.5 Higher order moments

In our experiments the PDF's actually carry the same information as the Wigner functions. Although the information is more conveniently displayed in the Wigner function, the PDF's can be more accurate than the Wigner functions due to the less complicated data processing.

We calculate the moments of different orders from the PDF's directly and the results are shown in Fig. 8.6. The 1st order moment μ_1 is the average of the quadrature amplitude. It is reasonably close to zero. The 2nd order moment, or variance (σ^2), is used to characterise the noise property of the light field (Fig. 8.6(b)). We get 6.8 dB of squeezing and 14.6 dB of anti-squeezing. $\alpha_3 = \mu_3/\sigma^3$ is defined as the skewness [140] of the PDF's, where μ_3 is the 3rd order moment. For a perfect Gaussian function, $\alpha_3 = 0$. We also have the kurtosis $\alpha_4 = \mu_4/\sigma^4$ [140], where μ_4 is the 4th order moment. $\alpha_4 = 3$ for a perfect Gaussian function. It's a characterisation of the "peakness" of the distribution. We know from Fig. 8.6(a) that our measured PDF's are not perfect Gaussian functions. This in turn gives the non-Gaussian feature of the reconstructed Wigner function. We have also analysed the PDF's of the vacuum state for calibration purpose. They have a much smaller skewness $\alpha_3 = -0.05$ and a comparable kurtosis $\alpha_4 = 6$.

For a sub-threshold OPO, the pump mode may be treated classically. The interaction Hamiltonian can be written as [83]

$$\hat{H}_I = \frac{i\hbar}{2}(\varepsilon \hat{a}^{\dagger 2} - \varepsilon^* \hat{a}^2) \tag{8.1}$$

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[2]The Wigner functions have been normalised by $2/\pi$.

where ε is proportional to the amplitude of the pump and the $\chi^{(2)}$ nonlinear susceptibility of the medium. The squeezed vacuum state thus generated should have a Gaussian Wigner function. (Notice that the symmetrical Gaussian Wigner function of a vacuum state is transformed into an asymmetrical Gaussian Wigner function.) This means that the odd order moments $\mu_3, \mu_5, \dots$ should all be zero.

However, the optical system doesn't have a stable enough phase reference. (The phase jitter over 90 seconds is estimated to be $1.5°$-$3°$ from the PDF's.) As visualisation we plot the standard deviations σ of the PDF's as a function of quadrature angle in the polar coordinate system in Fig. 8.7. The result differs from a smooth lemniscate [11] and suggests the stability of the phase should be improved in order to do more accurate OHT measurement. Because of this phase instability, we have to combine about ten traces of raw data together in order to get the PDF's. This process adds extra noise to the measured PDF's. We think the inaccuracy of this trace combination is the reason that we measured the higher order moments. We illustrate in Fig. 8.8 the effect that imperfect (non-centered) combination of Gaussian distributions gives a non-Gaussian distribution. In practice, for different scans of data acquisition, we always have slightly different degrees of squeezing. When we try to combine these scans together, due to the limited number of data points on each trace (32,768), we inevitably introduce inaccuracies. The combination can not be exactly centered. It's not surprising that this non-centered combination of different Gaussian distributions gives a non-Gaussian distribution. For better reconstruction, we suggest a shorter data acquisition time by using a high sampling rate I/O board, so that the inaccuracy due to the trace combination can be eliminated. In Fig. 8.9 we summarise the different technical steps which affect the Wigner function and show how an ideal Gaussian state evolves into a non-Gaussian state in the measurement/reconstruction process. (The Gaussian convolutions were discussed in section 8.2.3 and their mathematical details will be provided in the next section.) We conclude that the measured non-Gaussian features are due to the imperfect data combination.

8.3 The effect of electronic noise on the Wigner function of a squeezed vacuum state

In a realistic experiment of OHT, electronic noise is unavoidable. We want to discuss in this section how the electronic noise will affect the Wigner function of a squeezed vacuum state. We limit ourselves to the case of Gaussian electronic noise which agrees with measurements of the actual electronic noise in our experiments.

First of all let's study how two Gaussian distributions mix. Suppose we have two Gaussian distributions for two independent variables x_1 and x_2,

$$f_1(x_1) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{x_1^2}{2\sigma_1^2}\right), \tag{8.2}$$

$$f_2(x_2) = \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left(-\frac{x_2^2}{2\sigma_2^2}\right), \tag{8.3}$$

where σ_1^2, σ_2^2 are the variances for each distribution. Experimentally if we measure the sum variable $x = x_1 + x_2$, the distribution function of x will be

$$f(x) = \int dx_1 f_1(x_1) f_2(x - x_2)$$

$$= \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(-\frac{x^2}{2\sigma_0^2}\right), \tag{8.4}$$

where $\sigma_0^2 = \sigma_1^2 + \sigma_2^2$. Eq. (8.4) tells us the distribution $f(x)$ is the convolution of the individual distributions $f_1(x_1)$ and $f_2(x_2)$. We note that $f(x)$ is still a Gaussian distribution, however, with a larger variance which is the sum of the two individual variances σ_1^2 and σ_2^2. Now let's come back to the question of electronic noise in the measurement system. For a normalised coherent state, we have the PDF

$$w_c(x,\theta) = \frac{1}{\sqrt{2\pi}\sigma_c} \exp\left(-\frac{x^2}{2\sigma_c^2}\right), \tag{8.5}$$

where $\sigma_c = 1/2$. For a squeezed vacuum state, the corresponding PDF is

$$w_s(x,\theta) = \frac{1}{\sqrt{2\pi}\sigma_s(\theta)} \exp\left(-\frac{x^2}{2\sigma_s^2(\theta)}\right), \tag{8.6}$$

where the standard deviation

$$\sigma_s(\theta) = \sqrt{\sin^2(\theta)e^{-2r} + \cos^2(\theta)e^{2r}}/2 \tag{8.7}$$

is quadrature dependent. r is the squeezing parameter. For 6 dB of squeezing, $r \doteq 0.691$. $w_c(x,\theta)$ and $w_s(x,\theta)$ are the PDF's that will be measured in an ideal measurement system without electronic noise.

The Wigner function of a squeezed vacuum state is

$$W_s(p,q) = \frac{1}{2\pi\sigma_c^2} \exp[-(p^2 e^{-2r} + q^2 e^{2r})/(2\sigma_c^2)]. \tag{8.8}$$

Suppose we have a Gaussian electronic noise in our measurement system with the variance $\sigma_e^2 = \sigma_c^2/(2.3^2 - 1)$ (as estimated from our experimental data). In an OHT system, we measure the quadrature amplitude x which is the sum of the true quadrature amplitude x' and the electronic noise value x_e. According to Eq. (8.4) the measured PDF will be

$$w_{se}(x,\theta) = \frac{1}{\sqrt{2\pi}\sigma_{se}(\theta)} \exp\left[\frac{-x^2}{2\sigma_{se}^2(\theta)}\right], \tag{8.9}$$

where $\sigma_{se}(\theta) = \sqrt{\sigma_s^2(\theta) + \sigma_e^2}$ is the increased standard deviation. From $w_{se}(x,\theta)$ we can predict the reconstructed Wigner function to be

$$W_{se}(p,q) = \frac{1}{2\pi\sigma_{ce}^2} \exp[-(p^2 e^{-2r} + q^2 e^{2r})/(2\sigma_{ce}^2)]. \tag{8.10}$$

We show in Fig. 8.10 the standard variance $\sigma(\theta)$ as a function of quadrature angle θ (from 0 to $\pi/2$) with and without electronic noise.

From the discussion above we know that Gaussian electronic noise has a convolution effect on the Wigner function of a squeezed vacuum state, and the resulting Wigner function is a broadened Gaussian. This is the reason why we measured less squeezing (-4.1 dB) than in the conventional homodyne measurement (-7.0 dB). Due to the more complicated electronic configuration, the electronic noise of the OHT system is larger than that of the conventional homodyne measurement system.

8.4 Conclusions

In this chapter we studied OHT of a squeezed vacuum state which was generated from a degenerate sub-threshold optical parametric oscillator. The Wigner function of the squeezed vacuum state was reconstructed and calibrated by the corresponding vacuum state. We discussed the problem of phase instability and its effect on the reconstructed Wigner function. An alternative data acquisition/combination method was employed to avoid long term phase instability problem. Various sources of noise were discussed, such as the detection efficiency, electronic noise and numerical filtering. We provided a detailed analysis of the electronic noise. The noise sources have a convolution effect on the quantum state. In contrast to theoretical prediction, the reconstructed Wigner function was not a Gaussian. We analysed the resulting PDF's directly in detail and evaluated the higher order moments. (To the best of our knowledge, this is the first quantitative analysis of this kind.) Apart from the ability of OHT to visualise the squeezing feature in phase space qualitatively, there are limitations of realistic OHT. In particular it has not yet provided quantitatively satisfactory results. This shows that the accuracy of this technique must be improved in order to match the conventional homodyne detection. We addressed some possible reason for this imperfection and recommended improvements, specifically improved phase stability of the optical system, an increased optical local oscillator power for homodyning and a higher sampling rate I/O board for faster and more consistent data acquisition.

(a)

(b)

Figure 8.3: The reconstructed Wigner functions. (a) a squeezed vacuum state; (b) the corresponding vacuum state. green pump power: 145 mW; local oscillator power: 2 mW; detection frequency: 3 MHz; data points: 200,000.

Figure 8.4: Contour plot of the Wigner function of the squeezed vacuum state. squeezing: -4.1 dB; anti-squeezing: +8.7 dB (normalised to the vacuum state).

(a)

(b)

Figure 8.5: Measured squeezing and anti-squeezing (a) and calibrated uncertainty area (b) at different height. This figure shows the evaluation of the Wigner function in Fig. 8.4 at different contours ranging from 0.1 to 0.9 of the peak value. x: anti-squeezing; +: squeezing. The dotted horizontal lines represent +8.7 dB (anti-squeezing), 0 dB (quantum noise limit) and -4.1 dB (squeezing) respectively. The anti-squeezing (+8.7 dB) and squeezing (-4.1 dB) values are calculated at $1/e^2$ of the peak height.

Figure 8.6: Moments of the PDF's of the squeezed vacuum state. (a) +: mean value (1st order moment), ⋄: α_3 (skewness), ×: α_4 (kurtosis); (b) squeezing and anti-squeezing (2nd order moment, or variance) calibrated by the vacuum state. (Please refer to the text part for detailed explanation.)

Figure 8.7: Standard deviations of the probability density functions of a squeezed vacuum state in the polar coordinate system. The number along a radius represents standard deviations at the corresponding position.

Figure 8.8: The imperfect combination of two Gaussian distributions gives a non-Gaussian distribution. dotted: Gaussian distribution, $\sigma = 0.6$ (standard deviation), $\bar{x} = -0.05$ (average); dashdotted: Gaussian distribution, $\sigma = 0.62$, $\bar{x} = 0.05$; solid: average of the two Gaussian distributions results, which is a non-Gaussian distribution with non-zero higher order moments. ($\sigma \approx 0.61$, $\bar{x} = 0$, $\alpha_3 \approx 6.3 \times 10^{-3}$, $\alpha_4 \approx 2.9$.) The different value of σ accounts for different degree of squeezing of each scan, and the different value of $\bar{x}$ accounts for imperfect (non-centered) combination.

W ◄ − − Gaussian Wigner function of
the squeezed vacuum state

Gaussian electronic noise

W*G$_{el}$ ◄ − − Gaussian

finite detection efficiency

W*G$_{el}$*G$_{det}$ ◄ − − Gaussian

non-centered trace combination

W$_{rec}$ ◄ − − reconstructed Wigner
function; non-Gaussian

Figure 8.9: The Gaussian Wigner function of the squeezed vacuum state is convoluted by the Gaussian electronic noise and the Gaussian vacuum noise (due to the finite detection efficiency). Because of an imperfect (non-centered) trace combination, non-Gaussian features are introduced into the reconstructed Wigner function.

Figure 8.10: Standard variance $\sigma(\theta)$ as a function of the quadrature angle θ (from 0 to $\pi/2$). +: with electronic noise $\sigma_e^2 = \sigma_c^2/(2.3^2 - 1)$ (as estimated from our experimental data); *: without electronic noise.

Chapter 9

Producing quantum non-Gaussian states

9.1   Background and overview

The real strength of optical homodyne tomography is to reconstruct the non-Gaussian features of quantum states. In principle, standard homodyne measurement is good enough for measuring any quantum state which has Gaussian Wigner function [16], even if it's not a symmetric Gaussian state, such as a squeezed state. However, it's extremely difficult to generate a non-Gaussian optical state because a nonlinearity of order of at least $\chi^{(3)}$ is needed ($\chi^{(2)}$ nonlinear process can only create Gaussian optical states[1], see discussion in section 8.2.5). k-photon ($k \geq 3$) down-conversion has been extensively studied [141–147] and highly non-Gaussian Wigner functions were predicted. Three-photon down-conversion is the lowest-order process of this kind. The Wigner functions of three-photon down-conversion exhibit 3-fold symmetry and negativity. Other kinds of $\chi^{(3)}$ processes are the Kerr effect and cross-Kerr effect, they are also called self-phase modulation [148, 149] and cross-phase modulation respectively. The Kerr effect has been studied theoretically [150–152] and "banana-shaped" Wigner functions have been predicted. This is due to the intensity dependent phase modulation effect [150]. It was also pointed out [151, 152] that "Schrödinger's cat" states can be produced by Kerr effects. With the current technology of Kerr effect, it is impossible to achieve high enough nonlinearity in order to demonstrate the "Schrödinger's cat" states by the Kerr effect. On the other hand, the squeezing effect of Kerr media has been experimentally demonstrated [14, 153–162]. These correspond to the "banana-shape" stage of the Kerr effect. In the case of small $\chi^{(3)}$ nonlinearity, Kerr squeezing is similar to quadrature squeezing, which is due to $\chi^{(2)}$ nonlinearity. The squeezing in $\chi^{(3)}$ is due to the reduction of photon number uncertainty. As a penalty, the phase noise is enhanced[2].

Coudreau et. al. have demonstrated Kerr effect with cold atoms from a magneto-optical trap [14]. They even reconstructed the Wigner function of the laser beam which passed through the cold atoms. In principle, if the $\chi^{(3)}$ Kerr nonlinearity is large enough, they could have obtained a "banana-shaped" Wigner function. But this didn't happen in their experiment. They reconstructed an elliptic Wigner function, which is similar to the Wigner function of a quadrature squeezed state. This indicates the $\chi^{(3)}$ nonlinearity produced by their cold atoms was still very small.

It was proposed by Schmidt et al. [165] that a Giant Kerr nonlinearity can be obtained

---

[1]Spontaneous parametric down-conversion is a $\chi^{(2)}$ nonlinear process, and it creates conditional Fock states [25, 105–110], which are highly non-Gaussian. But with conditional measurement, it is consequently not a conventional $\chi^{(2)}$ process.

[2]Due to guided-acoustic-wave Brillouin scattering (GAWBS) [163, 164], which introduces noise from forward scattering of thermally excited guided acoustic modes, the phase noise can be much greater than what's imposed by the minimum uncertainty relationship.

by electromagnetically induced transparency (EIT), a quantum effect that permits the propagation of light pulses through an otherwise opaque medium. In a recent astonishing experiment utilising EIT, Hau et al. [166] have reduced the speed of light to 17 metres per second in an ultracold atomic gas. The giant Kerr nonlinearity was measured in their experiment and they obtained a nonlinear refractive index of 0.18 cm^2 W^{-1}, which is $\sim 10^6$ times greater than that measured in cold Cs atoms [167]. In the future this may lead to the experimental demonstration of "banana-shaped" or even "Schrödinger's cat" quantum states.

Cross-phase modulation is a less investigated area. Nevertheless squeezing has been observed by cross-phase modulation in semiconductors [168].

Cavity QED experiments have greater chance to produce non-Gaussian (or non-classical) states. In this kind of experiment, the coupling between atoms and cavity modes or applied laser beams can be large and well-controlled. "Schrödinger's cat" states [61, 62] and Fock states [21, 53, 54] have already been produced (see section 2.2).

In the rest of this chapter we focus on the optical field and present our methods of expressing the Wigner functions of the output states of self-phase modulation and cross-phase modulation. To the best of our knowledge, this work has not been included in any publication prior to this thesis.

9.2   Self-phase modulation

The interaction Hamiltonian of self-phase modulation is [169]

$$\hat{H}_I = \hbar\chi(\hat{a}^\dagger)^2\hat{a}^2, \tag{9.1}$$

where χ is the interaction parameter proportional to the third-order nonlinear susceptibility $\chi^{(3)}$ [170]. Suppose the input field is a coherent state

$$|\psi(0)\rangle = |\alpha\rangle. \tag{9.2}$$

After traveling in the Kerr medium of length L, the output state of the field can be written as

$$|\psi(t)\rangle = \hat{U}(L)|\alpha\rangle, \tag{9.3}$$

where $\hat{U}(L)$ is the unitary evolution operator. Suppose the field propagates at a velocity v, we can write down the evolution operator as

$$U(L) = \exp[\frac{i}{2}\lambda\hat{n}(\hat{n}-1)], \tag{9.4}$$

where

$$\lambda = \frac{2\chi L}{v} = 2\chi t. \tag{9.5}$$

We can expand the output state in the Fock representation,

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} \exp[\frac{i}{2}\lambda n(n-1)]\exp(-|\alpha|^2/2)\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{9.6}$$

The Q function of this output state was calculated by Kitagawa et al. [150] and the

result is

$$Q(\beta) = \frac{1}{\pi}e^{-(|\beta|^2+|\alpha|^2)}\Big|\sum_{n=0}^{\infty}\frac{(\beta^*\alpha)^n}{n!}e^{(i/2)\lambda n(n-1)}\Big|^2. \tag{9.7}$$

When

$$\lambda = 2\pi/m \tag{9.8}$$

and m is an integer, the Wigner functions were shown by Tara et al. [151]. Now we want to calculate the Wigner function of $|\psi(t)\rangle$ generally. Let's define s-parameterised displacement operators $\hat{D}(\beta,s)$ [65]

$$\hat{D}(\beta,s) \equiv \hat{D}(\beta)e^{s|\beta|^2/2} \tag{9.9}$$

and the complex Fourier transforms of the operators $\hat{D}(\beta,s)$,

$$\hat{T}(\beta,s) = \int \hat{D}(\xi,s)\exp(\beta\xi^* - \beta^*\xi)\frac{d^2\xi}{\pi}. \tag{9.10}$$

s-parameterised quasiprobability distribution (QPD) $W(\beta,s)$ is proportional to the expectation value of the operator $\hat{T}(\beta,s)$ [65],

$$W(\beta,s) = \frac{1}{\pi}Tr[\hat{\rho}\hat{T}(\beta,s)]. \tag{9.11}$$

When $s=0$, we get the Wigner function

$$
\begin{aligned}
W(\beta,0) &= \frac{1}{\pi}Tr[\hat{\rho}\hat{T}(\beta,0)] \\
&= \frac{1}{\pi}\sum_{n,m=0}^{\infty}\exp[\frac{i}{2}\lambda n(n-1) - \frac{i}{2}\lambda m(m-1) - |\alpha|^2]\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}} \\
&\quad \langle m|\hat{T}(\beta,0)|n\rangle.
\end{aligned}
\tag{9.12}
$$

The Fock representation matrix element $\langle m|\hat{T}(\beta,0)|n\rangle$ can be expressed as [66]

$$\langle m|\hat{T}(\beta,0)|n\rangle = (m!/n!)^{1/2}2^{n-m+1}(-1)^m(\beta^*)^{n-m}\exp(-2|\beta|^2)L_m^{(n-m)}(4|\beta|^2), \tag{9.13}$$

where $L_m^{(n-m)}(x)$ are the associated Laguerre polynomials. For $n \geq m$

$$L_m^{(n-m)}(x) = \sum_{l=0}^{m}(-1)^l\binom{n}{m-l}\frac{x^l}{l!}, \tag{9.14}$$

while for $n \leq m$,

$$L_m^{(n-m)}(x) = \sum_{l=0}^{n}\binom{n}{n-l}\frac{(-x)^{m-l}}{(m-l)!}. \tag{9.15}$$

Substituting Eq. (9.13) into Eq. (9.12), we can express the Wigner function by a serial sum

$$
\begin{aligned}
W(\beta,0) &= \frac{1}{\pi}\sum_{n,m=0}^{\infty}\exp[\frac{i}{2}\lambda n(n-1) - \frac{i}{2}\lambda m(m-1) - |\alpha|^2]\frac{\alpha^n\alpha^{*m}}{n!}2^{n-m-1}(-1)^m \\
&\quad (\beta^*)^{n-m}\exp(-2|\beta|^2)L_m^{(n-m)}(4|\beta|^2).
\end{aligned}
\tag{9.16}
$$

These output states are highly non-Gaussian and in general have negative values.

We show in Fig. 9.1 and Fig. 9.2 the contours of four Wigner functions with increasing λ from 0 to 0.15, the output state evolves from coherent state to "banana-shaped" state and "curved" state. With even larger $\lambda = -2\pi/1.5$ (i. e., $m = 1.5$ in Eq. (9.8)), we have a 3-component "Schrödinger's cat" state [152] which is shown in Fig. 9.3.

9.3 Cross-phase modulation

The interaction Hamiltonian of cross-phase modulation is [147]

$$\hat{H}_I = \hbar \kappa \hat{a}_1^\dagger \hat{a}_1 \hat{a}_3^\dagger \hat{a}_3, \tag{9.17}$$

where $\hat{a}_1$ and $\hat{a}_3$ are the annihilation operators of the fundamental wave (ω_1) and the third harmonic wave ($\omega_3 = 3\omega_1$) respectively. κ is the interaction parameter proportional to the third-order nonlinear susceptibility.

Suppose the input states of ω_1 and ω_3 waves are coherent,

$$|\psi(0)\rangle = |\alpha_1, \alpha_3\rangle. \tag{9.18}$$

The unitary evolution operator of this process is

$$\hat{U}(t) = \exp(-i\frac{\hat{H}}{\hbar}t) = \exp(-i\lambda \hat{a}_1^\dagger \hat{a}_1 \hat{a}_3^\dagger \hat{a}_3), \tag{9.19}$$

where $\lambda \equiv \kappa t$. We can expand the wave function of the output state in the Fock representation,

$$
\begin{aligned}
|\psi(t)\rangle &= \hat{U}(t)|\psi(0)\rangle \\
&= \exp(-i\lambda \hat{a}_1^\dagger \hat{a}_1 \hat{a}_3^\dagger \hat{a}_3)|\alpha_1, \alpha_3\rangle \\
&= \exp(-i\lambda \hat{n}_1 \hat{n}_3)|\alpha_1\rangle_1 |\alpha_3\rangle_3 \\
&= \exp(-i\lambda \hat{n}_1 \hat{n}_3) \sum_{n=0}^{\infty} \exp(-|\alpha_1|^2/2) \frac{\alpha_1^n}{\sqrt{n!}} |n\rangle_1 |\alpha_3\rangle_3 \\
&= \sum_{n=0}^{\infty} \exp(-|\alpha_1|^2/2) \frac{1}{\sqrt{n!}} \sum_{m=0}^{\infty} (\alpha_1 e^{-i\lambda \hat{n}_3})^n |n\rangle_1 e^{-|\alpha_3|^2/2} \frac{\alpha_3^m}{\sqrt{m!}} |m\rangle_3 \\
&= \sum_{m=0}^{\infty} \exp(-|\alpha_3|^2/2) \frac{1}{\sqrt{m!}} \alpha_3^m |m\rangle_3 |\alpha_1 e^{-i\lambda m}\rangle_1. \tag{9.20}
\end{aligned}
$$

In order to calculate the Wigner function of the fundamental wave ω_1, we take a trace over the third harmonic wave ω_3 and obtain the reduced density operator

$$
\begin{aligned}
\hat{\rho}_1(t) &= Tr_3[\hat{\rho}(t)] \\
&= \sum_{l=0}^{\infty} {}_3\langle l|\psi(t)\rangle\langle\psi(t)|l\rangle_3 \\
&= \sum_{l=0}^{\infty} \exp(-|\alpha_3|^2) \frac{|\alpha_3|^{2l}}{l!} |\alpha_1 e^{-i\lambda l}\rangle_1 \, {}_1\langle\alpha_1 e^{-i\lambda l}|. \tag{9.21}
\end{aligned}
$$

Using this operator identity,

$$\hat{T}(\beta, 0) = 2\hat{D}(\beta)(-1)^{\hat{a}^\dagger \hat{a}} \hat{D}^{-1}(\beta)$$

$$= \quad 2 : \exp[-2(\hat{a}^\dagger - \beta^*)(\hat{a} - \beta)] :, \tag{9.22}$$

we can write down the Wigner function of the output fundamental wave as

$$
\begin{aligned}
W(\beta, 0) &= Tr[\hat{\rho}_1(t)\hat{T}(\beta, 0)] \\
&= e^{-|\alpha_3|^2} \frac{|\alpha_3|^{2l}}{l!} {}_1\langle \alpha_1 e^{-i\lambda l} | 2 : \exp[-2(\hat{a}_1^\dagger - \beta^*)(\hat{a}_1 - \beta)] : |\alpha_1 e^{-i\lambda l}\rangle_1 \\
&= 2 \exp(-|\alpha_3|^2 - 2|\beta|^2 - 2|\alpha_1|^2) \sum_{l=0}^{\infty} \frac{|\alpha_3|^{2l}}{l!} e^{2(\beta\alpha_1^* e^{i\lambda l} + \beta^* \alpha_1 e^{-i\lambda l})}. \tag{9.23}
\end{aligned}
$$

These output states are highly non-Gaussian but do not have negative values. We show in Fig. 9.4 and Fig. 9.5 the contours of four Wigner functions with increasing λ from 0 to 0.15. The input states are $|\alpha_1\rangle = |1.0\rangle$ (fundamental wave ω_1) and $|\alpha_3\rangle = |10.0\rangle$ (third harmonic wave ω_3). With the increasing of λ, more non-Gaussian features can be produced.

9.4 Summary

In this chapter we addressed the practical difficulties that exist in producing non-Gaussian states of an optical field. We start with an overview of the various options that exist in quantum optics as well as cavity QED. While optical non-Gaussian states are extremely difficult to produce, there have been reports on the generation of non-Gaussian motional states of trapped ions and non-Gaussian states of a micromaser field. In the optical field the possible nonlinearities are those ones with higher order, at least $\chi^{(3)}$ nonlinearity. Using our Fock representation method we analyse one particular nonlinearity, namely self-phase modulation and cross-phase modulation. Given a sufficiently high nonlinearity these effects could produce quantum non-Gaussian features. The nonlinearities are orders of magnitudes larger than those obtained in present quantum noise experiment. However, in the light of the giant Kerr nonlinearity measured by Hau et al. [166] using a Bose-Einstein condensate (BEC) as the nonlinear medium one can predict that an experiment which produces non-Gaussian features is in principle feasible but extremely challenging. It would require the combination of squeezed state detection and BEC technology.

(a)

(b)

Figure 9.1: The Wigner function contours of the output states of self-phase modulation. The amplitude of the input coherent state is $\alpha = 4.0$. (a) $\lambda = 0$, coherent state; (b) $\lambda = 0.05$, "banana-shaped" state.

(a)

(b)

Figure 9.2: The Wigner function contours of the output states of self-phase modulation. The amplitude of the input coherent state is $\alpha = 4.0$. (a) $\lambda = 0.10$, "curved" state; (b) $\lambda = 0.15$, "curved" state.

(a)

(b)

Figure 9.3: The Wigner function (a) and its contour plot (b) of a 3-component "Schrödinger's cat" state produced by self-phase modulation; $\alpha = 2.0$, $\lambda = -2\pi/1.5$.

Figure 9.4: The Wigner function contours of the output states of cross-phase modulation. The amplitudes of the input coherent states are $\alpha_1 = 1.0$ and $\alpha_3 = 10.0$. (a) $\lambda = 0$, coherent state; (b) $\lambda = 0.05$.

Figure 9.5: The Wigner function contours of the output states of cross-phase modulation. The amplitudes of the input coherent states are $\alpha_1 = 1.0$ and $\alpha_3 = 10.0$. (a) $\lambda = 0.10$; (b) $\lambda = 0.15$.

Chapter 10

Conclusions

10.1   List of main results

The main results presented in this thesis can be summarised as follows:

(1) A semiclassical theory without linearisation of the quantum state was developed to model the optical homodyne tomography (OHT) system. This model explains the phase space correspondence of signal as well as noise. Asynchronous demodulation and such introduced phase quadrature diffusion can also be explained by the result of this general model. (see section 3.6.1)

(2) A way of simplifying the inverse Radon transform was given by means of the degenerated hypergeometric function. This method may save computing time for reconstructing Wigner functions from measured probability density functions (PDF's). (see section 3.5)

(3) OHT was implemented and used to measure the amplitude and phase noise of a diode pumped Nd:YAG laser. A high finesse optical cavity was constructed and employed to separate the laser noise from the quantum noise limited transmitted beam. These two beams were mixed when performing OHT. The amplitude and phase noise spectra were measured by conventional homodyning and phase noise was found to dominate in most of the RF regime. Pre-stabilisation of the laser was carried out and improvement in phase noise performance was recorded. Tomographic measurement was used and super-Poissonian elliptic Gaussian Wigner functions were reconstructed. In addition, phase quadrature diffusion was visualised in phase space. The interference problem of separated beams was explained. This problem was not encountered in normal homodyning. (see chapter 6)

(4) We investigated phase modulated coherent states. OHT turned out to be very useful in simultaneously determining the signal to noise ratio (SNR) in both the amplitude and the phase quadrature. By means of electronic techniques such as modulation-switching and phase-unlocking, we prepared classical non-Gaussian states and reconstructed their Wigner functions from experimental data. The successful reconstruction of non-Gaussian features through the measurement system assures the capability of OHT in the reconstruction of non-classical states such as "Schrödinger's cat" states. (see chapter 7)

(5) We measured the squeezed vacuum state and calibrated it with the corresponding vacuum state. Squeezing and anti-squeezing were calculated. Here the technical limitations of OHT began to play an important role. Both conventional homodyne detection and tomography are very susceptible to additional noise combinations. These can be non-Gaussian, as seen from our measured PDF's. Simply using a spectrum analyser these terms would have been ignored. The reconstructed Wigner function can have non-

Gaussian components which are not directly linked to the properties of the optical state. From the higher order moments analysis of the PDF's, we brought a link to the traditional method and assessed the abilities and limitations of realistic OHT. (see chapter 8)

(6) A problem with hysteresis of the PZT response was found and visualised by the measured PDF's. This drawback can also be regarded as an application of OHT for precise sensing. Based on this result, we implemented our OHT system by using precise optical phase from an interference trace. We believe this method wasn't used in other similar OHT experiments. (see section 4.2)

(7) A method based on Fock representation was developed to generally express the Wigner functions of the output states resulting from self-phase modulation and cross-phase modulation. (see chapter 9)

10.2   Outlook of optical homodyne tomography

As a powerful technique in quantum optics, OHT is still well worth experimental investigation. It has the ability to visualise the non-classical features of quantum states, such as the squeezing ellipse and negative value of the Wigner function. These are beyond the ability of the conventional homodyne measurements. Here we outline three possible future experiments.

The first one is the idea of producing a Fock state by means of conditionally detecting the signal wave in an optical parametric down-converter [171, 172]. We can first measure the presence of photons in one of the down-converted beams, and then use this signal to trigger the measurement of the other beam. Highly non-classical states may be produced and their Wigner functions reconstructed this way[1].

The second experiment is cascaded optical homodyning [31]. The simplicity of direct probing of phase space [26–28] together with the high quantum efficiency of optical homodyning make it quite attractive to do quantum state reconstruction using this method.

The third idea is to extend OHT to the multi-mode case, e. g., two-mode case. As pointed out by Tan et al. [173] and Banaszek et al. [174], beam splitting of a single photon can produce a two-mode state that can reveal non-locality. This is related to the two-mode Wigner function of the output state. In the case of single mode OHT, its strength relies on the visualisation of non-Gaussian features. However, in the two-mode case, it can also reveal entanglement, as was already shown by White et al. [25] by means of density matrix reconstruction.

In conclusion, we are of the firm belief that OHT will bring very interesting results in future experiments in quantum optics.

---

[1]During the revision process of this thesis, the quantum tomography of such conditional Fock state has been performed [111].

Appendix A

Programs for the reconstruction of the Wigner function

A.1   Program for data binning: mp125.c

```
/*Project: Quantum State Tomography.*/
/*Part No.: 1*/
/*Function: make probabilities (mp) from experimental data*/
/*Input data files: (from experiment)
quadrature noise value trace and photodetector DC trace
mt.dd
ctOriCrt.d (previous traces result, if any)
ctOriDrawCrt.d (previous traces result, if any)*/
/*Output data files:
param.d
ctOri.d
ctOriDraw.d
ctAng.d
wOri.d
wOriDraw.d*/
/*Author: Jinwei Wu; Email:jinwei_wu@yahoo.com*/

#include <stdio.h>
#include <stdlib.h>
#include <math.h>

/*general purpose*/
#define Pi (float) 3.14159
#define D1 (int) 230000 /*assuming data length*/

/*!!!when change these constants, please keep them the same as
in Part2!!!*/
#define xm (float) 1.0 /*maximum of x, for distribution*/
#define Nx (int) 41 /*number of bins of x*/
#define thetam (float) Pi /*maximum of theta*/
#define Ntheta (int) 120 /*number of angles*/
#define Nxx (int) 41 /*for pdf graph; number of bins of x*/
/*!!!when change these constants, please keep them the same as
in Part2!!!*/

main (int argc, char *argv[])
```

```
{

/*variables definitions*/
float wOri[2*Nx-1][Ntheta]; /*original probability distribution
                            function, Nx-1 is the center*/
int ctOri[2*Nx-1][Ntheta]; /*original counts for distribution,
                            Nx-1 is the center*/
int ctAng[Ntheta];/*counts for data belong to individual angles*/

float dtheta, dx; /*divisions*/

int D1m; /*real data length in the data file*/

float volt[D1]; /*DC trace voltage*/
float sx[D1]; /*signal x value, finally it will be multiplied
by multi*/
float amplitude; /*DC trace amplitude*/
float average; /*DC trace average*/
float theta; /*for storing the angle value from DC trace*/

float multi; /*multiplier for weak signal data (e.g.: 100)*/
float preDCoffset; /*estimated DC offset when making pdf's*/
float DCMin; /*DC trace minimum*/
float DCMax; /*DC trace maximum*/

/*for pdf graph*/
float wOriDraw[2*Nxx-1][Ntheta];/*probability distribution
                        function for drawing, Nxx-1 is the center*/
int ctOriDraw[2*Nxx-1][Ntheta];/*counts for distribution for
                                drawing, Nxx-1 is the center*/
float dxx;/*divisions*/

int i, j, k, l, n;
float r, s, t;
char ch;
FILE *fp1, *fp2;

/*Command help*/
if(argc !=2 )
{
printf("Please enter one input data file name after the
   command!\n");
printf("Please try again!\n");
exit(0);
};

/*read data file*/
if((fp1=fopen("mt.dd","r"))==NULL){
```

```
printf("Cannot open file. Please make sure you have mt.dd in the
    directory.\n");
printf("Format of mt.dd:\n");
printf("<multi value>\n");
printf("<preDCoffset value>\n");

exit(1);};

fscanf(fp1, "%f\n",  &multi);
fscanf(fp1, "%f\n",  &preDCoffset);

fclose(fp1);

if(multi<=0){
printf("Error: multi<=0.  Please check mt.dd\n");
exit(1);};

printf("multi=%f\n",multi);
printf("preDCoffset=%f\n",preDCoffset);

/*set the initial values to zero*/

D1m=(int)0;

for(i=0;i<=D1-1;i++)
{
volt[i]=(float)0;
sx[i]=(float)0;
};

/*read data file*/
if((fp1=fopen(argv[1],"r"))==NULL){
printf("Cannot open file.\n");
exit(1);};

i=0;
while(!feof(fp1)) {
fscanf(fp1, "%f\t%f\n",  &sx[i], &volt[i]);
i++;
};
fclose(fp1);

D1m=i;
printf("D1m=%d\n",D1m);

/*calculate DCMin, DCMax*/
DCMin=volt[0];
DCMax=volt[0];
```

```
for(i=1;i<=D1m-1;i++)
{
if(DCMin>volt[i]) DCMin=volt[i];
if(DCMax<volt[i]) DCMax=volt[i];
};
printf("DCMin=%g\n",DCMin);
printf("DCMax=%g\n",DCMax);

amplitude=(float)(DCMax-DCMin)/2;
average=(float)(DCMax+DCMin)/2;

/*deduce and multiply sx*/
for(i=0;i<=D1m-1;i++)
{
sx[i]=(sx[i]-preDCoffset)*multi;
};

/*read ctOri[2*Nx-1][Ntheta] data file*/

if((fp1=fopen("ctOri.d","r"))==NULL){
for(j=0;j<=Ntheta-1;j++)
{
for(i=0;i<=2*Nx-2;i++)
{
ctOri[i][j]=0;
};
};
fclose(fp1);
} else {
for(j=0;j<=Ntheta-1;j++)
{
for(i=0;i<=2*Nx-2;i++)
{
fscanf(fp1, "%d\n", &ctOri[i][j]);
};
};
fclose(fp1);
};
/*test*/
/*printf("Please enter a character, (period to exit, c to
continue):");
for(;;)
{
ch=getchar();
if(ch=='.') exit(0);
if(ch=='c') break;
};*/
```

```
/*read ctOriDraw[2*Nxx-1][Ntheta] data file*/

if((fp1=fopen("ctOriDraw.d","r"))==NULL){
for(j=0;j<=Ntheta-1;j++)
{
for(i=0;i<=2*Nxx-2;i++)
{
ctOriDraw[i][j]=0;
};
};
fclose(fp1);
} else {
for(j=0;j<=Ntheta-1;j++)
{
for(i=0;i<=2*Nxx-2;i++)
{
fscanf(fp1, "%d\n", &ctOriDraw[i][j]);
};
};
fclose(fp1);
};

/*update ctOri[2*Nx-1][Ntheta] from volt[i], sx[i]*/
dtheta=(float)thetam/Ntheta;
printf("dtheta=%g\n",dtheta);
dx=(float)xm/(Nx-1);
printf("dx=%g\n",dx);

for(i=0;i<=D1m-1;i++)
{
r=(volt[i]-average)/amplitude;
if(fabs(r)>1) {continue;
} else {
theta=acos(-r);
j=(int)floor(theta/dtheta);
k=(int)floor((sx[i]+xm)/dx);
}

if((k>=0)&&(k<=2*Nx-2)&&(j<=Ntheta-1))/*Now j will be
                                        0:Ntheta-1*/
{
ctOri[k][j]++;
};
};

/*update ctOriDraw[2*Nxx-1][Ntheta] from volt[i], sx[i]*/
dxx=(float)xm/(Nxx-1);
```

```c
printf("dxx=%g\n",dxx);

for(i=0;i<=D1m-1;i++)
{
r=(volt[i]-average)/amplitude;
if(fabs(r)>1) {continue;
} else {
theta=acos(-r);
j=(int)floor(theta/dtheta);
k=(int)floor((sx[i]+xm)/dxx);
}

if((k>=0)&&(k<=2*Nxx-2)&&(j<=Ntheta-1))/*Now j will be
                                          0:Ntheta-1*/
{
ctOriDraw[k][j]++;
};
};

/*write data file for parameters*/

if((fp1=fopen("param.d","w"))==NULL){
printf("Cannot open file.\n");
exit(1);};

fprintf(fp1, "DCMin=%f\n",DCMin);
fprintf(fp1, "DCMax=%f\n\n",DCMax);

fprintf(fp1, "D1m=%d\n",D1m);
fprintf(fp1, "dtheta=%g\n",dtheta);
fprintf(fp1, "dx=%g\n",dx);
fprintf(fp1, "dxx=%g\n\n",dxx);

fprintf(fp1, "multi=%f\n",multi);
fprintf(fp1, "preDCoffset=%f\n\n",preDCoffset);

fprintf(fp1, "==============\n\n");

fclose(fp1);

/*update ctOri[2*Nx-1][Ntheta] data file*/

if((fp1=fopen("ctOri.d","w"))==NULL){
printf("Cannot open file.\n");
exit(1);};

for(j=0;j<=Ntheta-1;j++)
{
```

```c
for(i=0;i<=2*Nx-2;i++)
{
fprintf(fp1, "%d\n", ctOri[i][j]);
};
};
fclose(fp1);

/*update ctOriDraw[2*Nxx-1][Ntheta] data file*/

if((fp1=fopen("ctOriDraw.d","w"))==NULL){
printf("Cannot open file.\n");
exit(1);};

for(j=0;j<=Ntheta-1;j++)
{
for(i=0;i<=2*Nxx-2;i++)
{
fprintf(fp1, "%d\n", ctOriDraw[i][j]);
};
};
fclose(fp1);

/*calculate wOri[2*Nx-1][Ntheta] from ctOri[2*Nx-1][Ntheta]*/
for(i=0;i<=Ntheta-1;i++)
{
ctAng[i]=0;
for(j=0;j<=2*Nx-2;j++)
{
ctAng[i]=ctAng[i]+ctOri[j][i];
};

if(ctAng[i]==0) {
for(j=0;j<=2*Nx-2;j++)
{
wOri[j][i]=0;
};
}
else {
for(j=0;j<=2*Nx-2;j++)
{
wOri[j][i]=(float)ctOri[j][i]/ctAng[i];
};
};
};

/*calculate wOriDraw[2*Nxx-1][Ntheta] from
ctOriDraw[2*Nxx-1][Ntheta]*/
for(i=0;i<=Ntheta-1;i++)
```

```
{
if(ctAng[i]==0) {
for(j=0;j<=2*Nxx-2;j++)
{
wOriDraw[j][i]=0;
};
}
else {
for(j=0;j<=2*Nxx-2;j++)
{
wOriDraw[j][i]=(float)ctOriDraw[j][i]/ctAng[i];
};
};
};


/*write data file*/

if((fp1=fopen("ctAng.d","w"))==NULL){
printf("Cannot open file.\n");
exit(1);};

for(j=0;j<=Ntheta-1;j++)
{
fprintf(fp1, "%d\n", ctAng[j]);
};
fclose(fp1);


/*write data file*/

if((fp1=fopen("wOri.d","w"))==NULL){
printf("Cannot open file.\n");
exit(1);};

for(j=0;j<=Ntheta-1;j++)
{
for(i=0;i<=2*Nx-2;i++)
{
fprintf(fp1, "%f\n", wOri[i][j]);
};
};
fclose(fp1);

/*write data file*/

if((fp1=fopen("wOriDraw.d","w"))==NULL){
printf("Cannot open file.\n");
exit(1);};
```

```
for(j=0;j<=Ntheta-1;j++)
{
for(i=0;i<=2*Nxx-2;i++)
{
fprintf(fp1, "%f\n", wOriDraw[i][j]);
};
};
fclose(fp1);

}/*End main*/
```

A.2   Program for the inverse Radon transform: tom125.c

```c
/*Project: Quantum State Tomography.*/
/*Part No.: 2*/
/*Function: Inverse Radon Transform (NB: tom -> tomography)*/
/*Input data files: (from mp.c output)
wOri.d
wOriDraw.d
nn.d
/*Output data files:
error.d (if any)
wOriNew.d
wOriDrawNew.d
param.d: parameters
wCenterOri.d
wCenterAng.d
wCenterFin.d
wr.d:reconstructed Wigner function*/
/*Author: Jinwei Wu; Email:jinwei_wu@yahoo.com*/

#include <stdio.h>
#include <stdlib.h>
#include <math.h>

/*general purpose*/
#define Pi (float) 3.14159
#define etam (float) 40 /*maximum of eta, should be very big
                           theoretically*/
#define Neta (int) 101 /*number of bins of eta*/
#define beta (float) 0.1 /*0.1*/ /*roll-off parameter*/
#define gama (float) beta*beta/(4*Pi) /*roll-off parameter*/

/*!!!when change these constants, please keep them the same as
in Part1!!!*/
#define xm (float) 1.0 /*maximum of x, for distribution*/
#define Nx (int) 41 /*number of bins of x*/
#define thetam (float) Pi /*maximum of theta*/
#define Ntheta (int) 120 /*number of angles*/
#define Nxx (int) 41 /*for pdf graph; number of bins of x*/
/*!!!when change these constants, please keep them the same as
in Part1!!!*/

/*for centered Wigner function*/
#define Npp (int) 41 /*number of bins of pp*/
#define Nqp (int) 41 /*number of bins of qp*/
#define ppm (float) 1.0 /*maximum of pp, for Wigner function*/
#define qpm (float) 1.0 /*maximum of qp, for Wigner function*/
#define xpm (float) 1.0  /*maximum of xp, for distribution*/
```

```
#define Nxp (int) 41 /*!!! Nxp-1=(Nx-1)*xpm/xm !!!, number of
                        bins of xp*/


main ()


{
/*variables definitions*/
float theta[Ntheta];
float x[2*Nx-1];/*Nx-1 is the center*/
float eta[Neta];
float wOri[2*Nx-1][Ntheta];/*original probability distribution
                            function, Nx-1 is the center*/
float wOriNew[2*Nx-1][Ntheta];/*new original probability
                    distribution function, Nx-1 is the center*/
float dtheta, dx, deta;/*divisions*/
float f_eta[Neta];/*function of eta in IRT*/
float f_cos[Ntheta],f_sin[Ntheta];/*cosine and sine functions*/
float eta_cos[Neta][Ntheta];/*eta[]*cos[]*/
float eta_sin[Neta][Ntheta];/*eta[]*sin[]*/


float factor;/*factor used for normalization of W*/
int n1;/*start index of pdf quadrature*/
int n2;/*end index of pdf quadrature*/
int NthetaNew;/*new number of angles*/


/*for centered Wigner function*/
float p0; /*pp center of Wigner*/
float q0; /*qp center of Wigner*/
float wCenter[2*Nxp-1][Ntheta];/*centered probability
                    distribution function, Nxp-1 is the center*/
float wCenterp[2*Nxp-1][Ntheta];/*probability distribution
    function, Nxp-1 is the center, for wCenter center seeking*/
float WCenter[2*Npp-1][2*Nqp-1];/*reconstructed Wigner function,
                            Npp-1:Nqp-1 is the center*/
float xp[2*Nxp-1]; /*Nxp-1 is the center*/
float pp[2*Npp-1]; /*Npp-1 is the center*/
float qp[2*Nqp-1]; /*Nqp-1 is the center*/


float dxp, dpp, dqp; /*divisions*/


float eta_xp[Neta][2*Nxp-1]; /*eta[]*xp[]*/


float wCenterAng[Ntheta]; /*total probability before
                            renormalization*/


float dcoffset; /*for correction of mixer trace DC offset due to
            amplifier noise or imperfect photodetector matching*/
float aveStart,aveEnd; /*average quadrature values for
```

```
theta=0 and theta=thetam-dtheta*/


/*for pdf graph*/
float wOriDraw[2*Nxx-1][Ntheta];/*probability distribution
                   function for drawing, Nxx-1 is the center*/
float wOriDrawNew[2*Nxx-1][Ntheta];/*new probability
      distribution function for drawing, Nxx-1 is the center*/
float dxx;/*divisions*/


int i, j, k, l, n;
float r, s, t;
char ch;
FILE *fp1, *fp2;


/*read data file*/
if((fp1=fopen("nn.d","r"))==NULL){
if((fp1=fopen("error.d","w"))==NULL){
printf("Cannot open file.\n");
exit(1);};
fprintf(fp1,"Cannot open file. Please make sure you have
nn.d in the directory.\n");
fprintf(fp1,"Format of nn.d:\n");
fprintf(fp1,"<n1 value>\n");
fprintf(fp1,"<n2 value>\n");
fclose(fp1);
exit(1);
};


fscanf(fp1, "%d\n",  &n1);
fscanf(fp1, "%d\n",  &n2);


fclose(fp1);


if(n1>Ntheta || n1<1 || n2>Ntheta || n2<1 || n1>=n2){
if((fp1=fopen("error.d","w"))==NULL){
printf("Cannot open file.\n");
exit(1);};
fprintf(fp1, "Error: n1 or n2 not appropriate.  Please
check nn.d\n");
fclose(fp1);
exit(1);
};


/*calculate divisions*/
NthetaNew=n2-n1+1;
dtheta=(float)thetam/NthetaNew;/*not NthetaNew-1 because we
                               don't want Pi*/
dx=(float)xm/(Nx-1);
```

```c
dxx=(float)xm/(Nxx-1);
dpp=(float)ppm/(Npp-1);
dqp=(float)qpm/(Nqp-1);
deta=(float)etam/(Neta-1);
dxp=dx;

/*preparing variables arrays*/

for(i=0;i<=NthetaNew-1;i++)
{
theta[i]=i*dtheta;
};

for(i=0;i<=2*Nx-2;i++)
{
x[i]=-xm+i*dx;
};


for(i=0;i<=2*Npp-2;i++)
{
pp[i]=-ppm+i*dpp;
};


for(i=0;i<=2*Nqp-2;i++)
{
qp[i]=-qpm+i*dqp;
};

for(i=0;i<=Neta-1;i++)
{
eta[i]=i*deta;
};

for(i=0;i<=2*Nxp-2;i++)
{
xp[i]=-xpm+i*dxp;
};


/*read data file for wOri*/
if((fp1=fopen("wOri.d","r"))==NULL){
printf("Cannot open file.\n");
exit(1);};

for(j=0;j<=Ntheta-1;j++)
{
```

```
for(i=0;i<=2*Nx-2;i++)
{
fscanf(fp1, "%f\n", &wOri[i][j]);
};
};
fclose(fp1);

/*calculate wOriNew from wOri*/
for(j=0;j<=NthetaNew-1;j++)
{
for(i=0;i<=2*Nx-2;i++)
{
wOriNew[i][j]=wOri[i][j+n1-1];
};
};

/*write data file*/
if((fp1=fopen("wOriNew.d","w"))==NULL){
printf("Cannot open file.\n");
exit(1);};

for(j=0;j<=NthetaNew-1;j++)
{
for(i=0;i<=2*Nx-2;i++)
{
fprintf(fp1, "%f\n", wOriNew[i][j]);
};
};
fclose(fp1);

/*read data file for wOriDraw*/
if((fp1=fopen("wOriDraw.d","r"))==NULL){
printf("Cannot open file.\n");
exit(1);};

for(j=0;j<=Ntheta-1;j++)
{
for(i=0;i<=2*Nxx-2;i++)
{
fscanf(fp1, "%f\n", &wOriDraw[i][j]);
};
};
fclose(fp1);

/*calculate wOriDrawNew from wOriDraw*/
for(j=0;j<=NthetaNew-1;j++)
{
for(i=0;i<=2*Nxx-2;i++)
```

```
{
wOriDrawNew[i][j]=wOriDraw[i][j+n1-1];
};
};

/*write data file*/
if((fp1=fopen("wOriDrawNew.d","w"))==NULL){
printf("Cannot open file.\n");
exit(1);};

for(j=0;j<=NthetaNew-1;j++)
{
for(i=0;i<=2*Nxx-2;i++)
{
fprintf(fp1, "%f\n", wOriDrawNew[i][j]);
};
};
fclose(fp1);

/*calculate aveStart,aveEnd*/
aveStart=0;
aveEnd=0;
for(i=0;i<=2*Nx-2;i++) {
aveStart=(float)aveStart+i*wOriNew[i][0];
aveEnd=(float)aveEnd+i*wOriNew[i][NthetaNew-1];
};
aveStart=aveStart*dx-xm;
aveEnd=aveEnd*dx-xm;
dcoffset=(aveStart+aveEnd)/2;

k=floor(dcoffset/dx);
/*printf("k=%d\n",k);*/

/*calculate p0,q0 while deducting dcoffset*/
p0=0;
q0=0;
r=(float)NthetaNew/2;
j=floor(r);
for(i=0;i<=2*Nx-2;i++) {
p0=(float)p0+i*wOriNew[i][0];
q0=(float)q0+i*wOriNew[i][j];
};
p0=p0-k;
q0=q0-k;
p0=p0*dx-xm;
q0=q0*dx-xm;

/*write data file for parameters*/
```

```
if((fp1=fopen("param.d","a"))==NULL){
printf("Cannot open file.\n");
exit(1);};

fprintf(fp1, "p0= %g\n", p0);
fprintf(fp1, "q0= %g\n", q0);
fprintf(fp1, "dcoffset= %g\n\n",dcoffset);

fprintf(fp1, "etam= %f\n", etam);
fprintf(fp1, "Neta= %d\n", Neta);
fprintf(fp1, "beta= %f\n\n", beta);

fprintf(fp1, "xm= %f\n", xm);
fprintf(fp1, "Nx= %d\n", Nx);
fprintf(fp1, "thetam= %f\n", thetam);
fprintf(fp1, "Ntheta= %d\n", Ntheta);
fprintf(fp1, "NthetaNew= %d\n", NthetaNew);
fprintf(fp1, "Nxx= %d\n", Nxx);
fprintf(fp1, "n1= %d\n", n1);
fprintf(fp1, "n2= %d\n\n", n2);

fprintf(fp1, "Npp= %d\n", Npp);
fprintf(fp1, "Nqp= %d\n", Nqp);
fprintf(fp1, "ppm= %f\n", ppm);
fprintf(fp1, "qpm= %f\n", qpm);
fprintf(fp1, "xpm= %f\n", xpm);
fprintf(fp1, "Nxp= %d\n", Nxp);

fprintf(fp1, "==============\n\n");

fclose(fp1);

/*preparing intermediate data for IRT*/
for(l=0;l<=Neta-1;l++) {
f_eta[l]=eta[l]*exp(-gama*eta[l]*eta[l]);
};

for(n=0;n<=NthetaNew-1;n++) {
f_cos[n]=cos(theta[n]);
f_sin[n]=sin(theta[n]);
};

for(l=0;l<=Neta-1;l++) {
for(k=0;k<=2*Nxp-2;k++) {
eta_xp[l][k]=eta[l]*xp[k];
};
};
```

```
for(l=0;l<=Neta-1;l++) {
for(n=0;n<=NthetaNew-1;n++) {
eta_cos[l][n]=eta[l]*f_cos[n];
eta_sin[l][n]=eta[l]*f_sin[n];
};
};


/*calculate wCenter from wOriNew*/
for(j=0;j<=NthetaNew-1;j++)
{
r=p0*f_cos[j]+q0*f_sin[j]-xpm;/*correspondent x[k] of -xpm*/
k=floor((r+dcoffset+xm)/dx);

for(i=0;i<=2*Nxp-2;i++)
{
if((k>=0)&&(k<=2*Nx-2)) {
wCenter[i][j]=wOriNew[k][j];
} else {
wCenter[i][j]=0.0;/*out of the range, set to 0.0*/
};
k++;
};

};


/*write data file for wCenter*/
if((fp1=fopen("wCenterOri.d","w"))==NULL){
printf("Cannot open file.\n");
exit(1);};

for(j=0;j<=NthetaNew-1;j++)
{
for(i=0;i<=2*Nxp-2;i++)
{
fprintf(fp1, "%f\n", wCenter[i][j]);
};
};
fclose(fp1);


/*center-seeking pdf wCenter, for unbalance problem*/
for(i=0;i<=NthetaNew-1;i++)
{
r=0;
for(j=0;j<=2*Nxp-2;j++)
{
r=(float)r+wCenter[j][i]*j;/*center of wCenter[][]*/
};
k=floor(r-(Nxp-1));
```

```
for(j=0;j<=2*Nxp-2;j++)
{
if((j+k>=0)&&(j+k<=2*Nxp-2)) {
wCenterp[j][i]=wCenter[j+k][i];
} else {
wCenterp[j][i]=0;
};
};


};


/*transfer wCenterp back to wCenter*/
for(i=0;i<=NthetaNew-1;i++)
{
for(j=0;j<=2*Nxp-2;j++)
{
wCenter[j][i]=wCenterp[j][i];
};
};


/*renormalize the pdf wCenter*/
for(i=0;i<=NthetaNew-1;i++)
{
wCenterAng[i]=0;
for(j=0;j<=2*Nxp-2;j++)
{
wCenterAng[i]=wCenterAng[i]+wCenter[j][i];
};

if(wCenterAng[i]==0) {
continue;
}
else {
for(j=0;j<=2*Nxp-2;j++)
{
wCenter[j][i]=(float)wCenter[j][i]/wCenterAng[i];
};
};


};


/*write data file for wCenterAng*/
if((fp1=fopen("wCenterAng.d","w"))==NULL){
printf("Cannot open file.\n");
exit(1);};
```

```
for(j=0;j<=NthetaNew-1;j++)
{
fprintf(fp1, "%f\n", wCenterAng[j]);
};
fclose(fp1);




/*write data file for wCenter*/
if((fp1=fopen("wCenterFin.d","w"))==NULL){
printf("Cannot open file.\n");
exit(1);};

for(j=0;j<=NthetaNew-1;j++)
{
for(i=0;i<=2*Nxp-2;i++)
{
fprintf(fp1, "%f\n", wCenter[i][j]);
};
};
fclose(fp1);

/*calculate WCenter from wCenter*/
factor=deta*dxp*dtheta/(2*Pi*Pi);
for(i=0;i<=2*Npp-2;i++)
{
for(j=0;j<=2*Nqp-2;j++)
{

WCenter[i][j]=(float)0;
for(k=0;k<=2*Nxp-2;k++)
{
for(l=0;l<=Neta-1;l++)
{
for(n=0;n<=NthetaNew-1;n++)
{
WCenter[i][j]=WCenter[i][j]+
   wCenter[k][n]*f_eta[l]*cos(eta_xp[l][k]-pp[i]*eta_cos[l][n]
   -qp[j]*eta_sin[l][n]);
};
};
};
WCenter[i][j]=WCenter[i][j]*factor;

};
};

/*write data file for reconstructed Wigner function*/
```

```
if((fp1=fopen("wr.d","w"))==NULL){
printf("Cannot open file.\n");
exit(1);};

for(i=0;i<=2*Npp-2;i++)
{
for(j=0;j<=2*Nqp-2;j++)
{
fprintf(fp1, "%f\n", WCenter[i][j]);
};
};
fclose(fp1);

}/*End main*/
```

Appendix B

Direct probing of quantum phase space

As an alternative of optical homodyne tomography (OHT) for quantum state reconstruction (QSR), direct probing of quantum phase space is numerically much simpler [26–29]. It doesn't require any integration for the reconstruction of Wigner function.

In this Appendix we will summarise the basic theory of this method. For details please refer to Ref. [27]. As discussed in section 9.2, the s-parameterised quasiprobability distribution (QPD) $W(\alpha, s)$ is proportional to the expectation value of the operator $\hat{T}(\alpha, s)$ [65],

$$W(\alpha, s) = \frac{1}{\pi} Tr[\hat{\rho}\hat{T}(\alpha, s)]. \tag{B.1}$$

The operator $\hat{T}(\alpha, s)$ is the complex Fourier transform of the operator $\hat{D}(\alpha, s)$,

$$\hat{T}(\alpha, s) = \int \hat{D}(\xi, s) \exp(\alpha\xi^* - \alpha^*\xi) \frac{d^2\xi}{\pi}, \tag{B.2}$$

and $\hat{D}(\alpha, s)$ is the s-parameterised displacement operator,

$$\begin{aligned} \hat{D}(\alpha, s) &\equiv e^{s|\alpha|^2/2} \\ &= \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})e^{s|\alpha|^2/2}. \end{aligned} \tag{B.3}$$

Here $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators of the optical field respectively. $\hat{T}(\alpha, s)$ can be expressed as a normally ordered operator,

$$\hat{T}(\alpha, s) = \frac{2}{1-s} : \exp(-\frac{2}{1-s}(\alpha^* - \hat{a}^\dagger)(\alpha - \hat{a})) : \tag{B.4}$$

where : : denotes the normal ordering of operators. When $s = 0$, from Eq. (B.4) we can calculate $\hat{T}(0, 0)$,

$$\begin{aligned} \hat{T}(0, 0) &= 2 : \exp(-2\hat{a}^\dagger\hat{a}) : \\ &= 2 \sum_{n=0}^{\infty} (-1)^n : e^{-\hat{a}^\dagger\hat{a}} \frac{(\hat{a}^\dagger\hat{a})^n}{n!} : \\ &= 2 \sum_{n=0}^{\infty} (-1)^n |n\rangle\langle n|. \end{aligned} \tag{B.5}$$

We know that $W(\alpha, 0) \equiv W(\alpha)$ is the Wigner function. From Eq. (B.5) we can obtain the value of the Wigner function at the origin of phase space,

$$W(0) = \frac{1}{\pi}\langle\hat{T}(0, 0)\rangle = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n p_n, \tag{B.6}$$

where p_n is the probability of counting n photons by an ideal photon counter. If we can somehow scan the phase space and shift the origin of phase space around, Eq. (B.6) provides a way to fully reconstruct the Wigner function.

Figure B.1: Experimental setup for the direct probing of quantum phase space. $\hat{a}_s$, $\hat{a}_p$, $\hat{a}_{out}$: annihilation operators of the signal wave, the probe wave and the output wave respectively; BS: beam splitter; PC: photon counter.

Shown in Fig. B.1 is a realisation of this idea. This scheme is named as direct probing of quantum phase space. $\hat{a}_s$, $\hat{a}_p$ and $\hat{a}_{out}$ are the annihilation operators of the signal wave, the probe wave and the output wave respectively. $\hat{a}_{out}$ can be written as a linear combination of $\hat{a}_s$ and $\hat{a}_p$,

$$\hat{a}_{out} = \sqrt{T}\hat{a}_s - \sqrt{1-T}\hat{a}_p \tag{B.7}$$

where T is the power transmittance of the beam splitter. Using Eq. (B.4) and Eq. (B.7), we can evaluate $\hat{T}_{out}(0;0)$ of the output wave in terms of the incoming waves,

$$
\begin{aligned}
\hat{T}_{out}(0;0) &= 2 : \exp(-2\hat{a}_{out}^\dagger \hat{a}_{out}) : \\
&= 2 : \exp\left\{-2T[\hat{a}_s^\dagger - \sqrt{(1-T)/T}\hat{a}_p^\dagger][\hat{a}_s - \sqrt{(1-T)/T}\hat{a}_p]\right\} : .
\end{aligned}
\tag{B.8}
$$

Suppose the probe wave is a coherent state $|\alpha\rangle$, from Eq. (B.1), Eq. (B.4) and Eq. (B.8) we can readily obtain a relationship between the QPD's of the output wave and the signal wave,

$$W_{out}(0) = \frac{1}{T}W_s\left(\sqrt{\frac{1-T}{T}}\alpha; -\frac{1-T}{T}\right), \tag{B.9}$$

where $W_{out}(0)$ is the Wigner function of the output wave at the origin of phase space and $W_s(\sqrt{\frac{1-T}{T}}\alpha; -\frac{1-T}{T})$ is the $s = 1 - 1/T$ ordered QPD of the signal wave at $\sqrt{\frac{1-T}{T}}\alpha$. According to Eq. (B.6), $W_{out}(0)$ can be determined by photon counting of the output wave. Based on this, Eq. (B.9) tells us that the QPD of the signal wave can be fully determined by scanning the probe wave amplitude and phase, i. e., the complex value α and measuring the photon counting statistics of the output wave. When the power transmittance T approaches unity, Eq. (B.9) provides a measurement of the Wigner function of the signal wave.

This method is numerically very simple, no integration is needed. However, it requires two-fold scanning (the amplitude and the phase of the probe wave). In OHT only the phase of the local oscillator needs to be scanned. One drawback of the direct probing method is that the quantum efficiency of photon counters (70% for avalanche photodi-

odes operated in the Geiger mode) is normally much lower than that of photodiodes (94% for InGaAS photodiodes operated at 1064 nm). Another drawback of the direct probing method is that the extra noise in the probe wave can not be canceled out as in a balanced homodyne scheme.

Appendix C

Circuit diagrams

Figure C.1: Circuit diagram of a low noise photodetector. [Courtesy of M. B. Gray, the Australian national university.]

L1

+1.5 V

C1 C2 C5 C7

Input

R1

4 + 8 R2 Output

1 7

5 2 HA5002

L2

-1.5 V

C3 C4 C6 C8

R1: 200 Ω; R1: 50 Ω;   C1, C3: 33 μF;
L1, L2: 10 μH;   C2, C4, C5, C6, C7, C8: 100 nF

Figure C.2: Circuit diagram of a buffer amplifier.

R1, R3: 1 kΩ; R2: 20 kΩ; P4: 10 Ω; C1, C3: 33 µF;
L1, L2: 10 µH;   C2, C4, C5, C6, C7, C8: 100 nF

Figure C.3: Circuit diagram of a signal amplifier.

Appendix D

PCI-MIO-16XE-50 I/O Connector

Shown in Fig. D.1 is the I/O connector of the PCI-MIO-16XE-50 multifunction I/O board for PCI. Please refer to Table D.1 for a description of the signal inputs. In our optical homodyne tomography (OHT) experiment, we normally need to acquire two channels of data, one for the quadrature amplitude and the other for the optical phase signal. In this case, we use (34, 68) as one analog channel (ACH0) and (31, 65) as the other channel (ACH2). If three analog channels are needed, we can also use (26, 60) as another channel (ACH5). These are all bipolar differential channels. Care should be taken when connecting these channels. Make sure all the acquired signal share the same common ground with the I/O board, otherwise nasty oscillation may occur.

Signal Names	Signal Description
ACH <>	Analog input channels
DIO <>, PA <>, PB <>	Digital input/output lines
PFI <>	Programmable inputs for trigger, conversion, and clock signals
AISENSE	Common signal input for NRSE analog inputs
DAC0OUT, DAC1OUT	Analog output channels
EXTREF	Analog output external reference voltage input
SCANCLK	Multiplexer scan clock for driving SCXI and AMUX-64T
EXTSTROBE*	External multiplexer strobe for driving SCXI and AMUX-64T
GPCTR_GATE	General purpose counter/timer gate input
GPCTR_OUT	General purpose counter/timer output
GPCTR_SOURCE	General purpose counter/timer input
FREQ_OUT	Programmable frequency output

Table D.1: I/O signal connection description (from National Instruments Inc. catalog).

ACH8 —	34	68	— ACH0
ACH1 —	33	67	— AIGND
AIGND —	32	66	— ACH9
ACH10 —	31	65	— ACH2
ACH3 —	30	64	— AIGND
AIGND —	29	63	— ACH11
ACH4 —	28	62	— AISENSE
AIGND —	27	61	— ACH12
ACH13 —	26	60	— ACH5
ACH6 —	25	59	— AIGND
AIGND —	24	58	— ACH14
ACH15 —	23	57	— ACH7
DAC0OUT —	22	56	— AIGND
DAC1OUT —	21	55	— AOGND
UNASSIGNED —	20	54	— AOGND
DIO4 —	19	53	— DGND
DGND —	18	52	— DIO0
DIO1 —	17	51	— DIO5
DIO6 —	16	50	— DGND
DGND —	15	49	— DIO2
5 V —	14	48	— DIO7
DGND —	13	47	— DIO3
DGND —	12	46	— SCANCLK
PFI0/TRIG1 —	11	45	— EXTSTROBE*
PFI1/TRIG2 —	10	44	— DGND
DGND —	9	43	— PFI2/CONVERT*
5 V —	8	42	— PFI3/GPCTR1_SOURCE
DGND —	7	41	— PFI4/GPCTR1_GATE
PFI5/UPDATE* —	6	40	— GPCTR1_OUT
PFI6/WFTRIG —	5	39	— DGND
DGND —	4	38	— PFI7/STARTSCAN
PFI9/GPCTR0_GATE —	3	37	— PFI8/GPCTR0_SOURCE
GPCTR0_OUT —	2	36	— DGND
FREQ_OUT —	1	35	— DGND

Figure D.1: PCI-MIO-16XE-50 I/O Connector (from National Instruments Inc. catalog).

Appendix E

Produce phase and amplitude modulations simultaneously

In this chapter we want to discuss how to produce phase and amplitude modulations simultaneously by an electro-optic modulator. The optical configuration is shown in Fig. E.1 and the different axes of the KDP crystal are illustrated in Fig. E.2.

Figure E.1: An electro-optic modulator which can produce phase and amplitude modulations simultaneously. The angle between x' and x is adjusted by rotating the crystal. (Slightly modified from Fig. 14.4 in Ref. [1].)

As shown in Fig. E.2, $\hat{x}', \hat{y}'$ are the electrically induced birefringent axes of the KDP crystal, $(\hat{x}', \hat{y}')$ and $(\hat{x}_1, \hat{y}_1)$ are fixed together and the angle between $\hat{y}'$ and $\hat{y}_1$ is $\pi/4$. The output polariser is in the direction of $\hat{y}_1$. The input beam is linearly polarised in the direction of $\hat{x}$, so the input electric field

$$\vec{E}_{in} = \sqrt{2}E_0 \cos(\omega t)\hat{x}. \tag{E.1}$$

From Fig. E.2 we know

$$\hat{x} = \cos(\alpha)\hat{x}' + \sin(\alpha)\hat{y}', \tag{E.2}$$

so $\vec{E}_{in}$ can be written as

$$\vec{E}_{in} = \sqrt{2}E_0(\cos(\alpha)\hat{x}' + \sin(\alpha)\hat{y}') \cos(\omega t). \tag{E.3}$$

Equivalently we have the projections of $\vec{E}_{in}$ along the $\hat{x}'$ and $\hat{y}'$ directions,

$$\begin{aligned} E_{x'}(0) &= \sqrt{2}E_0 \cos(\alpha), \\ E_{y'}(0) &= \sqrt{2}E_0 \sin(\alpha). \end{aligned} \tag{E.4}$$

Figure E.2: Different axes of a KDP crystal. $(x_1 \perp y_1)$ are the natural optical axes, $(x' \perp y')$ are the electrically induced birefringent axes and the angle between x' and x_1 is fixed to $\pi/4$. The input and output polarisers are in the directions of $x(\perp y)$ and y_1 respectively and $\otimes$ is the direction of the laser beam.

On the output surface of the crystal, the electric fields along the $\hat{x}'$ and $\hat{y}'$ directions are (note that we have ignored a common phase factor which is of no importance.)

$$
\begin{aligned}
E_{x'}(l) &= \sqrt{2}E_0 \cos(\alpha), \\
E_{y'}(l) &= \sqrt{2}E_0 \sin(\alpha)e^{-i\Gamma}.
\end{aligned}
\tag{E.5}
$$

From Eq. (E.5) we can write down the output field of the output polariser as

$$
(E_{y1})_o = [E_{y'}(l)/\sqrt{2} - E_{x'}(l)/\sqrt{2}]e^{i\omega t}.
\tag{E.6}
$$

Here $\Gamma = \pi/2 + \Gamma_m \sin(\Omega_m t)$. $\Gamma_m \ll 1$ is the modulation index and Ω_m is the modulation frequency. From Eq. (E.6) we know the output intensity

$$
\begin{aligned}
I_o &\propto (E_{y1})_o(E_{y1})_o^* \\
&= E_0^2(1 - \sin(2\alpha)\cos(\Gamma)).
\end{aligned}
\tag{E.7}
$$

When $\Gamma \approx \pi/2$, which is the case we are discussing, $I_o = E_0^2$ is independent of α.

Let's denote the real part of the output field as

$$
\begin{aligned}
E_{y1} &= Re\{(E_{y1})_o\} \\
&= E_0 Im\{[-\beta\sin(\alpha)(e^{i\Omega_m t} - e^{-i\Omega_m t}) - \cos(\alpha)e^{i\pi/2} + \sin(\alpha)]e^{i\omega t}\}.
\end{aligned}
\tag{E.8}
$$

We can write down the corresponding annihilation operator of this electric field

$$
\begin{aligned}
\hat{a}(t) &\leftrightarrow E_0\{-\beta\sin(\alpha)[e^{i\Omega_m t} - e^{-i\Omega_m t}] - \cos(\alpha)e^{i\pi/2} + \sin(\alpha)\} \\
&= E_0\{e^{-i(\pi/2-\alpha)} + \beta\sin(\alpha)[e^{-i\Omega_m t} - e^{i\Omega_m t}]\},
\end{aligned}
\tag{E.9}
$$

where $\beta \equiv J_1(\Gamma_m)$ and $\beta \ll 1$. $J_1(x)$ is the first order Bessel function. In order to include the quantum fluctuations of the input field, we replace E_0 by $E_0 + \delta\hat{a}(t)$ and rewrite the annihilation operator $\hat{a}(t)$ as

$$
\begin{aligned}
\hat{a}(t) &= [E_0 + \delta\hat{a}(t)]\{e^{i(\alpha - \pi/2)} + \beta\sin(\alpha)[e^{-i\Omega_m t} - e^{i\Omega_m t}]\} \\
&\approx -i\{E_0 e^{i\alpha} + 2\beta E_0 \sin(\alpha)\sin(\Omega_m t) + e^{i\alpha}\delta\hat{a}(t)\}.
\end{aligned}
\tag{E.10}
$$

Here we have assumed βE_0 and $\delta\hat{a}(t)$ to be very small comparing with E_0 and omitted the term containing $\beta E_0 \delta\hat{a}(t)$. We treat the strong local oscillator beam classically and approximate its annihilation operator by a c-number

$$
\hat{b}(t) \approx B e^{i\theta},
\tag{E.11}
$$

where θ is the local oscillator phase, which is scanned by a PZT. For a balanced homodyne detection, the difference current can be expressed by the field operators,

$$
\hat{i}_-(\theta, t) = \hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}.
\tag{E.12}
$$

The amplitude and phase quadrature operators can be defined as

$$
\begin{aligned}
\hat{X} &= Re\{\hat{a}\} = (\hat{a} + \hat{a}^\dagger)/2, \\
\hat{Y} &= Im\{\hat{a}\} = (\hat{a} - \hat{a}^\dagger)/(2i).
\end{aligned}
\tag{E.13}
$$

We can further define the rotated quadrature operator

$$
\hat{X}(\theta) = \hat{X}\cos(\theta) + \hat{Y}\sin(\theta).
\tag{E.14}
$$

Substituting Eq. (E.10), Eq. (E.13) and Eq. (E.14) into Eq. (E.12) we have

$$
\begin{aligned}
\hat{i}_-(\theta, t) &= 2B\hat{X}(\theta, t) \\
&= 2B[E_0 \sin(\alpha - \theta) - 2\beta E_0 \sin(\alpha)\sin(\theta)\sin(\Omega_m t) \\
&\quad + \delta\hat{X}(\pi/2 - \alpha + \theta, t)].
\end{aligned}
\tag{E.15}
$$

The local oscillator current of the mixer is

$$
i_l(t) \approx \sin(\Omega_d t + \psi),
\tag{E.16}
$$

where ψ is the electronic phase delay angle. The output current of the mixer is the product of its two input signals,

$$
i_M(\theta, t) = i_-(\theta, t)i_l(t).
\tag{E.17}
$$

Let's define the Fourier transform as

$$
\begin{aligned}
\tilde{y}(\Omega') &\equiv F[y(t)] \\
&= \int_{-\infty}^{+\infty} y(t)e^{-i\Omega' t}dt.
\end{aligned}
\tag{E.18}
$$

The Fourier amplitude $\tilde{i}_M(\theta, \Omega')$ at sideband Ω' is the convolution of the Fourier transforms of the input signals,

$$
\tilde{i}_M(\theta, \Omega') = \tilde{i}_-(\theta, \Omega') \star \tilde{i}_l(\Omega')
$$

$$= \int_{-\infty}^{+\infty} \tilde{i}_-(\theta, \lambda)\tilde{i}_l(\Omega' - \lambda)d\lambda. \qquad (E.19)$$

Now the output current of the low-pass filter can be expressed as

$$
\begin{aligned}
i_{\Omega d}(\theta, t) &= LPF[i_M(\theta, t)] = \frac{1}{2\pi} \int_{-\Delta\Omega_f}^{\Delta\Omega_f} \tilde{i}_M(\theta, \Omega')e^{j\Omega' t}d\Omega' \\
&= -\frac{j}{2}e^{j\psi}\left\{2\pi j\beta E_0 \sin(\alpha) \sin(\theta)\left[-\int_{-\Delta\Omega_f}^{\Delta\Omega_f} \delta(\Omega' - \Omega_d + \Omega_m)e^{j\Omega' t}d\Omega'\right]\right\} \\
&\quad +\frac{j}{2}e^{-j\psi}\left\{2\pi j\beta E_0 \sin(\alpha) \sin(\theta)\left[\int_{-\Delta\Omega_f}^{\Delta\Omega_f} \delta(\Omega' + \Omega_d - \Omega_m)e^{j\Omega' t}d\Omega'\right]\right\} \\
&\quad -\frac{j}{2}e^{j\psi}\int_{-\Delta\Omega_f}^{\Delta\Omega_f} \delta\tilde{X}(\pi/2 - \alpha + \theta, \Omega' - \Omega_d)e^{j\Omega' t}d\Omega' \\
&\quad +\frac{j}{2}e^{-j\psi}\int_{-\Delta\Omega_f}^{\Delta\Omega_f} \delta\tilde{X}(\pi/2 - \alpha + \theta, \Omega' + \Omega_d)e^{j\Omega' t}d\Omega', \qquad (E.20)
\end{aligned}
$$

where $\Delta\Omega_f$ is the bandwidth of the low-pass filter, which is about 50 kHz in our experiment. Note that we have assumed $\Omega_d \gg \Delta\Omega_f$ and $\Omega_m \gg \Delta\Omega_f$, which is always true in the experimental case. If we define this finite-bandwidth time domain function

$$\delta\hat{X}_c(\pi/2 - \alpha + \theta, -\Omega_d; t) = \frac{1}{2\pi} \int_{-\Delta\Omega_f}^{\Delta\Omega_f} \delta\tilde{X}(\pi/2 - \alpha + \theta, \Omega' - \Omega_d)e^{j\Omega' t}d\Omega', \qquad (E.21)$$

we can prove that

$$\delta\hat{X}_c^\dagger(\pi/2 - \alpha + \theta, -\Omega_d; t) = \delta\hat{X}_c(\pi/2 - \alpha + \theta, \Omega_d; t). \qquad (E.22)$$

Thus we have two Hermitian operators

$$
\begin{aligned}
\delta\hat{X}_{cr}(\pi/2 - \alpha + \theta, \Omega_d; t) &\equiv (\delta\hat{X}_c(\pi/2 - \alpha + \theta, \Omega; t) + \delta\hat{X}_c(\pi/2 - \alpha + \theta, -\Omega_d; t))/2, \\
\delta\hat{X}_{ci}(\pi/2 - \alpha + \theta, \Omega_d; t) &\equiv (\delta\hat{X}_c(\pi/2 - \alpha + \theta, \Omega; t) \\
&\quad - \delta\hat{X}_c(\pi/2 - \alpha + \theta, -\Omega_d; t))/(2j). \qquad (E.23)
\end{aligned}
$$

Substituting Eq. (E.21) and Eq. (E.23) into Eq. (E.20), we find

$$
\begin{aligned}
i_{\Omega d}(\theta, t) &= -2\pi\beta E_0 \sin(\alpha) \sin(\theta)\delta(\Omega_d, \Omega_m) + 2\pi \sin(\psi)\delta\hat{X}_{cr}(\pi/2 - \alpha + \theta, \Omega_d; t) \\
&\quad - 2\pi \cos(\psi)\delta\hat{X}_{ci}(\pi/2 - \alpha + \theta, \Omega_d; t), \qquad (E.24)
\end{aligned}
$$

where

$$\delta(\Omega_d, \Omega_m) \equiv \int_{-\Delta\Omega_f}^{\Delta\Omega_f} \delta(\Omega' - \Omega_d + \Omega_m) \cos(\Omega' t + \psi)d\Omega'. \qquad (E.25)$$

is a function defined for convenience.

$$\delta(\Omega_d, \Omega_m) = \cos[\psi + (\Omega_d - \Omega_m)t] \qquad (E.26)$$

when $|\Omega_d - \Omega_m| < \Delta\Omega_f$ and

$$\delta(\Omega_d, \Omega_m) = 0 \qquad (E.27)$$

when $|\Omega_d - \Omega_m| > \Delta\Omega_f$. $\delta\hat{X}_c(\pi/2 - \alpha + \theta, \Omega_d; t)$ can be understood as the total quantum fluctuation centered around $\pm\Omega_d$. $i_{\Omega d}(\theta, t)$ is the final electronic signal we measure in the

experiment. For a synchronous detection, $\Omega_d = \Omega_m$ and $\delta(\Omega_d, \Omega_m) = \cos(\psi)$. For in phase detection, $\psi = 0$, $i_l(t) \propto \sin(\Omega_d t) = \sin(\Omega_m t)$,

$$i_{\Omega d}(\theta, t) = -2\pi\beta E_0 \sin(\alpha)\sin(\theta) - 2\pi\delta\hat{X}_{ci}(\pi/2 - \alpha + \theta, \Omega_d; t). \tag{E.28}$$

The first term in Eq. (E.28) is due to the modulation. In the phase space, it gives us a displacement from the origin. For in quadrature detection, $\psi = \pi/2$, $i_l(t) \propto \cos(\Omega_d t) = \cos(\Omega_m t)$,

$$i_{\Omega d}(\theta, t) = 2\pi\delta\hat{X}_{cr}(\pi/2 - \alpha + \theta, \Omega_d; t). \tag{E.29}$$

There is no displacement in the phase space and the Wigner function is centered around the origin.

Generally, from Eq. (E.6),

$$\begin{aligned}(E_{y1})_o &= E_0[\sin(\alpha)e^{-j\Gamma} - \cos(\alpha)]e^{j\omega t} \\ &\approx -E_0(1 + \beta\sin(2\alpha)\sin(\Omega_m t))e^{j\omega t + j(\alpha - 2\beta\sin^2(\alpha)\sin(\Omega_m t))}.\end{aligned} \tag{E.30}$$

There are both phase and amplitude modulations.

For synchronous and in-phase detection, we list some examples of the output field $(E_{y1})_o$, output current $i_{\Omega d}(\theta, t)$ and the corresponding phase-space contours. Notice that the $-\delta x_{ci}(\pi/2)$ quadrature corresponds to $\theta = \alpha$.

(i) $\alpha = 0$:

$$(E_{y1})_o = -E_0 e^{j\omega t}, \tag{E.31}$$

$$i_{\Omega d}(\theta, t) = -2\pi\delta\hat{X}_{ci}(\pi/2 + \theta, \Omega_d; t). \tag{E.32}$$

$-\delta x_{ci}(\pi/2)$ $(\theta=0)$

$-\delta x_{ci}(0)$

AM (amplitude modulation): none
PM (phase modulation): none

Figure E.3: Phase space contour plot.

(ii) $\alpha = \pi/4$:

$$(E_{y1})_o = -E_0(1 + \beta\sin(\Omega_m t))e^{j\omega t + j(\pi/4 - \beta\sin(\Omega_m t))}, \tag{E.33}$$

$$i_{\Omega d}(\theta, t) = -\sqrt{2}\pi\beta E_0 \sin(\theta) - 2\pi\delta\hat{X}_{ci}(\pi/4 + \theta, \Omega_d; t). \tag{E.34}$$

-δx$_{ci}$(π/2) (θ=π/4)

π/4

-δx$_{ci}$(0)

$\sqrt{2}\,\pi\beta E_0$

AM: maximal
PM: medium

Figure E.4: Phase space contour plot.

(iii) $\alpha = \pi/2$:

$$(E_{y1})_o = -E_0 e^{j\omega t + j(\pi/2 - 2\beta\sin(\Omega_m t))}, \tag{E.35}$$

$$i_{\Omega d}(\theta, t) = -2\pi\beta E_0 \sin(\theta) - 2\pi\delta\hat{X}_{ci}(\theta, \Omega_d; t). \tag{E.36}$$

(iv) $\alpha = 3\pi/4$:

$$(E_{y1})_o = -E_0(1 - \beta\sin(\Omega_m t))e^{j\omega t + j(3\pi/4 - \beta\sin(\Omega_m t))}, \tag{E.37}$$

$$i_{\Omega d}(\theta, t) = -\sqrt{2}\pi\beta E_0 \sin(\theta) - 2\pi\delta\hat{X}_{ci}(\theta - \pi/4, \Omega_d; t). \tag{E.38}$$

(v) $\alpha = \pi$:

$$(E_{y1})_o = -E_0 e^{j\omega t + j\pi}, \tag{E.39}$$

$$i_{\Omega d}(\theta, t) = -2\pi\delta\hat{X}_{ci}(\theta - \pi/2, \Omega_d; t). \tag{E.40}$$

Figure E.5: Phase space contour plot.

Figure E.6: Phase space contour plot.

$-\delta x_{ci}(\pi/2)$ $(\theta{=}\pi)$

$-\delta x_{ci}(0)$

AM: none
PM: none

Figure E.7: Phase space contour plot.

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